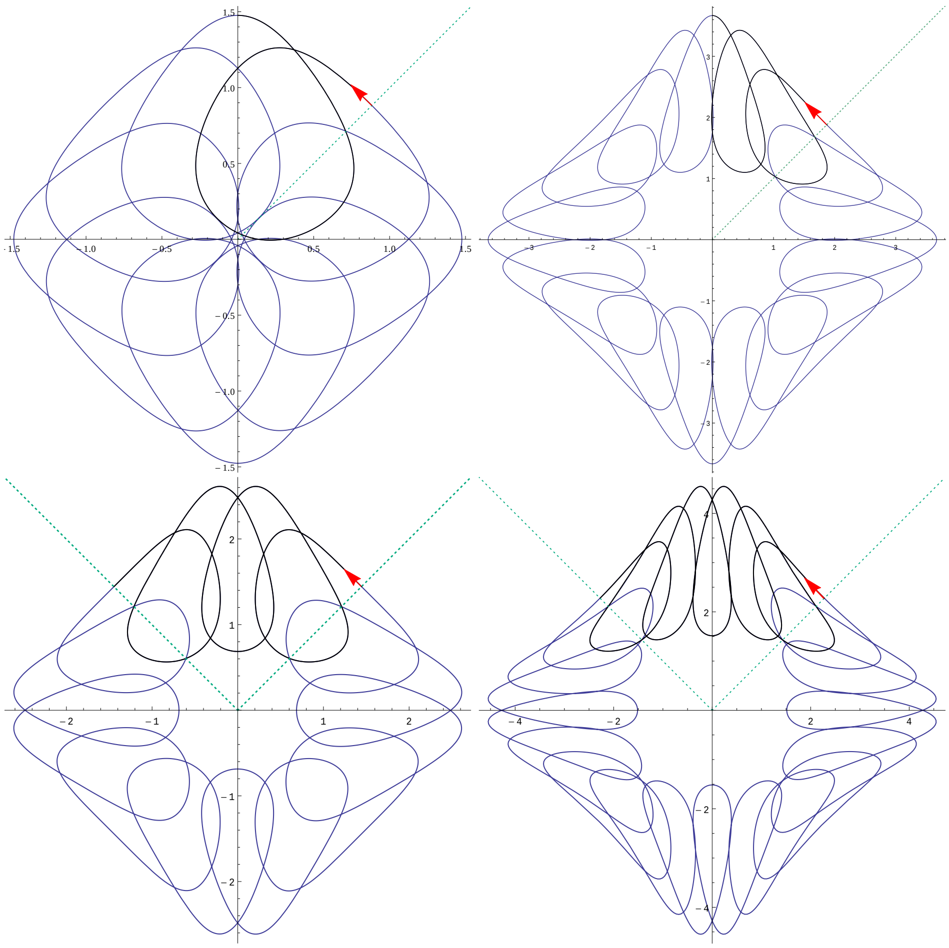
CONSTANT MEAN CURVATURE ANNULI IN HOMOGENEOUS MANIFOLDS

MIROSLAV VRŽINA



CONSTANT MEAN CURVATURE ANNULI IN HOMOGENEOUS MANIFOLDS

VOM FACHBEREICH MATHEMATIK
DER TECHNISCHEN UNIVERSITÄT DARMSTADT
ZUR ERLANGUNG DES GRADES EINES
DOKTORS DER NATURWISSENSCHAFTEN (DR. RER. NAT)
GENEHMIGTE DISSERTATION

VON
DIPL.-MATH. MIROSLAV VRŽINA
AUS OFFENBACH AM MAIN

REFERENT:	PROF. DR. KARSTEN GROSSE-BRAUCKMANN
KORREFERENT:	PRIVATDOZENT DR. MATTHIAS SCHNEIDER
TAG DER EINREICHUNG:	5. FEBRUAR 2016
TAG DER MÜNDLICHEN PRÜFUNG:	22. APRIL 2016

DARMSTADT 2016
D17

Danksagung

An dieser Stelle möchte ich einigen Personen danken, welche dafür verantwortlich sind, dass ich diese Promotionszeit in guter Erinnerung behalten werde.

Zunächst möchte ich mich bei meinem Doktorvater Karsten Große-Brauckmann bedanken. Karsten, Du hast mich ermutigt, den in dieser Arbeit diskutierten Fragen nachzugehen und Du warst dabei stets für Diskussionen verfügbar. Die Betreuung und Zusammenarbeit mit Dir waren prima, sei es in Forschung oder Lehre. Dafür, vor allem für das intensive Lesen der Arbeit vor der Abgabe, bin ich Dir sehr dankbar! Es hat sich gelohnt noch ein paar „Iterationen"durchzuführen. Du hast mir auch ermöglicht, Rob Kusner Ergebnisse aus der Arbeit vorzustellen, was nicht nur sehr hilfreich, sondern auch ganz angenehm war. Thank you, Rob, for taking your time, for the discussions, and for your anecdotes!

Weiter bedanke ich mich bei Privatdozent Matthias Schneider für die Übernahme des Korreferats, sowie bei Prof. Anna von Pippich und Prof. Mads Kyed für die Bereitschaft als Prüfer bei der Verteidigung zu fungieren.

Über die nun fast fünf Jahre bin ich vom dienstjüngsten zum dienstältesten Mitglied der Arbeitsgruppe Geometrie und Approximation „aufgestiegen". Egal in welcher Besetzung, es herrschte immer eine gute Atmosphäre in der AG, sodass wir den (an einigen Stellen absurden) Uni-Alltag gut bewältigen konnten. Die fast täglichen Doppelkopfrunden mit Kollegen und Studenten sollen hier nicht unerwähnt bleiben. Vielen Dank für die gute Zeit und bei manchen auch für das Korrekturlesen der Einleitung. Ich hoffe, eure Geduld (beim Warten darauf, dass ich eine Karte lege) noch ein weiteres halbes Jahr zu strapazieren.

Schließlich danke ich noch meiner Familie und meinen Freunden, dass ich auch privat den Kopf frei kriegen konnte. Insbesondere meine Nichten und Neffen haben hier ganze Arbeit geleistet. Und ja, in den letzten fünf Monaten vor der Abgabe hatte ich sicherlich weniger Zeit als sonst für euch alle, was sich nun bessern sollte.

Na kraju posebno se zahvaljujem mojim roditeljima. Hvala na svemu što ste meni ućinili i pomogli!

Zusammenfassung

In dieser Arbeit konstruieren wir Kreisringe konstanter mittlerer Krümmung in homogenen 3-Mannigfaltigkeiten. Diese Kreisringe verallgemeinern Zylinder und Unduloide aus $\mathbb{R}^3$. Ein Unduloid in $\mathbb{R}^3$ ist eine Rotationsfläche konstanter mittlerer Krümmung, welche einfach periodisch ist bezüglich Translationen längs der Rotationsachse. In homogenen 3-Mannigfaltigkeiten, zum Beispiel in dem Produkt $\mathbb{H}^2 \times \mathbb{R}$ aus hyperbolischer Ebene $\mathbb{H}^2$ und reeller Achse $\mathbb{R}$, sind zwar Translationen längs beliebiger Geodätischer Isometrien, aber Rotationen sind nur um Geodätische in bestimmter Lage isometrisch. Dennoch vermutet man, dass beliebige Unduloide, z.B. in $\mathbb{H}^2 \times \mathbb{R}$, existieren; es handelt sich dann um einfach periodische Kreisringe konstanter mittlerer Krümmung. Mit vertikaler und horizontaler Achse gibt es entsprechende Zylinder und Unduloide in $\mathbb{H}^2 \times \mathbb{R}$, aber der Fall einer geneigten Achse ist bislang nicht untersucht worden.

Im ersten Teil der Arbeit betrachten wir translationsinvariante Flächen in $\mathbb{H}^2 \times \mathbb{R}$ und in weiteren homogenen 3-Mannigfaltigkeiten, um dort Zylinder konstanter mittlerer Krümmung zu konstruieren. In diesem Fall reduziert sich die Gleichung für konstante mittlere Krümmung auf eine gewöhnliche Differentialgleichung. In den genannten Räumen ist diese Differentialgleichung komplizierter und analytische Lösungsansätze scheitern. Wir verfolgen einen geometrischen Ansatz: Wir betrachten invariante Flächen konstanter mittlerer Krümmung, die von Graphen erzeugt werden. Ein Vergleich mit Sphären konstanter mittlerer Krümmung zeigt, dass der Graph zu einer einfach geschlossenen Kurve fortgesetzt werden kann. Dadurch erhalten wir geneigte Zylinder in $\mathbb{H}^2 \times \mathbb{R}$ und eine Vielzahl weiterer Zylinder in anderen Räumen, zum Beispiel in Sol_3 und $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$.

Der zweite Teil betrifft die Konstruktion einfach periodischer Kreisringe konstanter mittlerer Krümmung in $\mathbb{H}^2 \times \mathbb{R}$. Dieser Fall führt im Allgemeinen, zum Beispiel bei einer geneigten Achse, auf eine partielle Differentialgleichung und zwar als freies Randwertproblem. Die Daniel Korrespondenz reduziert dieses freie Randwertproblem auf ein festes Randwertproblem in einer kompakten 3-Mannigfaltigkeit M. Wir zeigen, dass die bekannten Beispiele (vertikales und horizontales Unduloid) unter dieser Korrespondenz minimalen Kreisringen in M entsprechen, die von verschlungenen geodätischen Kreisen in M berandet werden. Anschließend beweisen wir, dass verschlungene geodätische Kreise mindestens zwei minimale Kreisringe beranden. Unter der Annahme, dass es genau zwei minimale Kreisringe gibt, können wir geodätische Kreise in M angeben, die zu geneigten Unduloiden in $\mathbb{H}^2 \times \mathbb{R}$ führen. Diese Eindeutigkeitsfrage in M verbleibt aber als offenes Problem.

Contents

Introduction

In this introduction we first give an overview of constant mean curvature surfaces. Our focus is on fundamental existence and uniqueness results which naturally explain the importance of the problems considered in the present thesis. We then give a detailed description of the structure and the main results obtained in this thesis.

Constant mean curvature surfaces in $\mathbb{R}^3$

Surfaces of constant mean curvature have a long and rich history in mathematics. In real life we encounter them as soap bubbles or soap films. In the 19th century, the Belgian physicist Joseph Plateau (1801 – 1883) carried out experiments with soap films and other liquids. He dipped wire frameworks into soap water and obtained soap films spanned by closed wires. The corresponding mathematical problem is to find a surface with least area among all surfaces bounded by a closed curve. This problem is known as *the Plateau problem*.

The concept of *mean curvature* is important for the mathematical interpretation of soap films. Mean curvature was first introduced in 1831 by Sophie Germain (1776 – 1831). It can be considered a force per area, that is, a pressure. For instance, a soap bubble separates gases or liquids with ambient pressures acting on the soap film. If the bubble remains in equilibrium the ambient pressure difference must be constant. Geometrically, mean curvature of a surface can be described as follows: Let p be a point on a surface $\Sigma \subset \mathbb{R}^3$ and consider a plane containing the line perpendicular to Σ at p. This plane intersects Σ in a planar curve, to which we assign the curvature κ with respect to the normal. Rotating the plane about the normal line through p we obtain a minimal and a maximal curvature, the so-called *principal curvatures* κ_1 and κ_2.

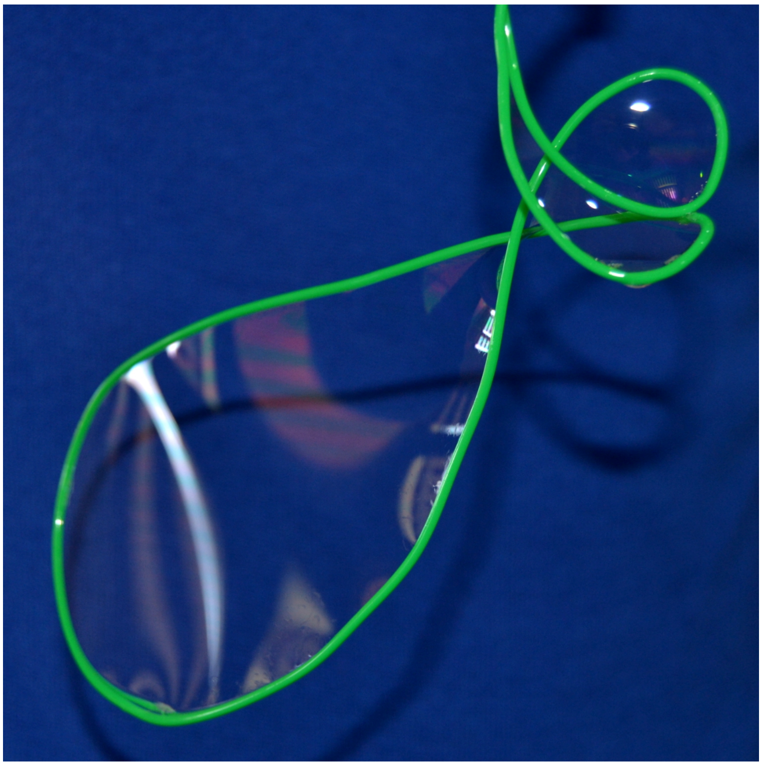


FIGURE A. Plateau problem: Soap film spanned by a closed wire

The arithmetic mean

$$H = \frac{\kappa_1 + \kappa_2}{2}$$

is called *mean curvature* of Σ. For instance, a round sphere $\mathbb{S}_r := \{p \in \mathbb{R}^3 \colon |p| = r\}$ with radius $r > 0$ has constant mean curvature $H = 1/r$.

Notation. *Let H be a real number. We call a surface Σ with constant mean curvature equal to $H \neq 0$ an* MCH-surface. *In case $H \equiv 0$ we call Σ a* minimal surface.

We move on to prominent examples of MCH-surfaces, namely, constant mean curvature surfaces of revolution. These were completely classified by Delaunay (1816 – 1872) in 1841. To formulate the problem rigorously, let

$$(r, h) \colon I \to (0, \infty) \times \mathbb{R}$$

be a regular curve. We place it in the x-z-plane and rotate it about the z-axis in $\mathbb{R}^3$. Then the principal curvatures are

$$\kappa_1 = \frac{r'h'' - h'r''}{\sqrt{r'^2 + h'^2}^3} \quad \text{and} \quad \kappa_2 = \frac{1}{r}\frac{h'}{\sqrt{r'^2 + h'^2}}.$$

We observe that κ_1 is the curvature of (r, h) and that κ_2 is the curvature of a circle whenever $r' = 0$. Requiring the mean curvature $H = \frac{\kappa_1+\kappa_2}{2}$ to be constant leads to an ordinary differential equation (ODE) the curve (r, h) must satisfy. In the papers [**Del41, Stu41**], Delaunay and Sturm showed that a solution (r, h) for $H \neq 0$ is a *roulette* of a focus of the conic sections with

$$\text{long semi-axis } a = \frac{1}{|H|} \quad \text{and} \quad \text{small semi-axis } b \leq a.$$

These conic sections are ellipses or hyperbolas. In the limiting case $H \to 0$ we obtain a parabola. A roulette is obtained as follows: we let a conic section in the x-z-plane touch the z-axis tangentially and "roll" it along the z-axis. Then the focus of the conic section traces out a curve (r, h) which solves the above ODE. The roulette of a parabola is a *catenary* and it generates a *catenoid* (case $H \equiv 0$). The roulette of an ellipse is an embedded curve. It is useful to consider the extremal cases of the ellipse: the focus of a circle (case $a = b$ above) traces out a line and the surface generated is a cylinder, whereas in the case $b = 0$ the ellipse degenerates to a segment and the roulette generates a chain of spheres touching each other tangentially. For a general ellipse we obtain a curve that is periodic with respect to discrete translations along the z-axis and the surface it generates is called *unduloid*; see Figure B. The roulette of a hyperbola is also periodic along the z-axis, but has self-intersections; it generates the so-called *nodoids*.

These shapes arise in the following experiment of Plateau: Take a cylindrical pipe and dip it into soap water on one end. Blow up the soap film from the other end and cap that end off so that we end up with part of a soap bubble bounded by one end of the pipe. We can now catch the soap bubble with a second pipe (again capping one end off). Keeping the pipes axially symmetric we see a bulge of an unduloid, see Figure D, and pulling them further apart we – at one point – obtain the shape of a cylinder.

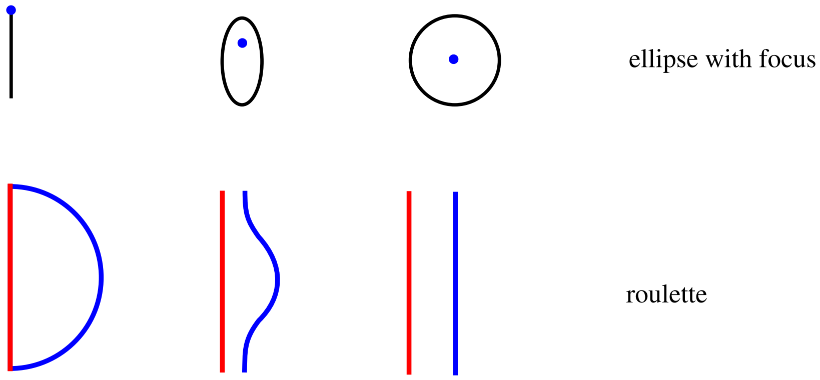


FIGURE B. Fundamental piece of a roulette of the focus of an ellipse; the focus of the ellipse and its trace are coloured in blue, the axis of rotation is coloured in red.

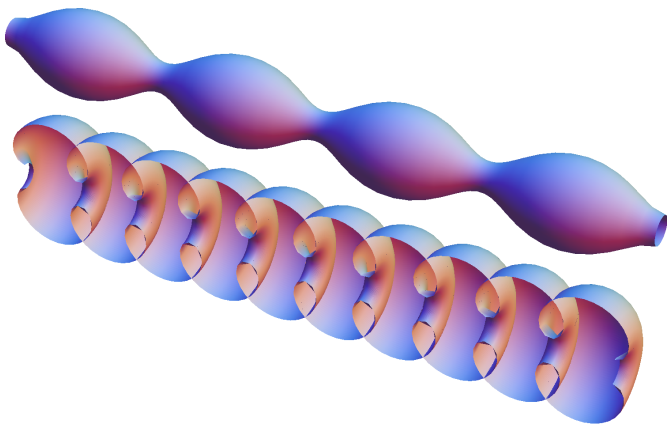


FIGURE C. An unduloid and a view of half a nodoid

In the 20th century many efforts were devoted to solve Plateau's problem: Given a Jordan curve Γ in $\mathbb{R}^3$, is there a surface of least area among all surfaces bounded by Γ? A solution of this problem necessarily is a minimal surface. The minimisation problem was solved independently by Jesse Douglas in 1931 and Tibor Radó in 1933. They used direct methods in the calculus of variations.

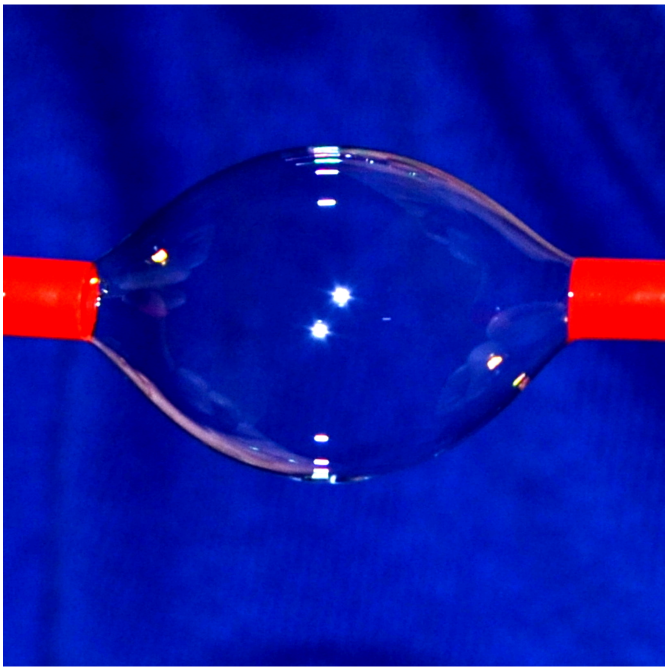


FIGURE D. Soap film experiment yielding a bulge of an unduloid

An MCH-surface can be considered a minimiser of the area functional under a volume constraint. The Plateau problem for MCH-surfaces thus can be formulated as follows: Given a Jordan curve Γ in $\mathbb{R}^3$ and a real number H, can we find an MCH-surface bounded by Γ? This question has an affirmative answer due to Hildebrandt in 1970 for H with $|H| \leq 1$. In case $|H| > 1$ it can be proved that the circle

$$\Gamma = \{(\cos(\varphi), \sin(\varphi), 0) \colon 0 \leq \varphi < 2\pi\}$$

admits no solution. This shows that construction of MCH-surfaces is more difficult than the construction of minimal surfaces.

Another difference between minimal and MCH-surfaces in $\mathbb{R}^3$ is the fact that there are no compact minimal surfaces. In 1956 Alexandrov showed that soap bubbles must be round. In Alexandrov's theorem geometry implies topology:

Theorem (Alexandrov, 1956)**.** *Let Σ be an embedded compact* MCH*-surface in $\mathbb{R}^3$. Then Σ is a round sphere of radius $\frac{1}{H}$, that is, $\Sigma = \mathbb{S}_{1/H}$.*

Alexandrov's method to prove this theorem is known as *moving planes argument* or *Alexandrov's reflection principle*.

If we fix the topology to that of a sphere we can weaken the assumptions:

Theorem (Hopf, 1957)**.** *Let Σ be an immersed* MCH*-surface in $\mathbb{R}^3$ homeomorphic to a sphere. Then Σ is a round sphere.*

The method of proof for Hopf's theorem is quite different. Hopf introduced a holomorphic quadratic differential. Complex analysis and topology imply that an immersed MCH-sphere is a round sphere.

In this sense soap bubbles are unique. The quoted theorems indicate that geometric conditions imposed on a (topological) surface have strong implications. Thus it is natural to ask: What are the minimal surfaces or MCH-surfaces homeomorphic to an annulus/cylinder? In 1983 Schoen studied this problem for $H \equiv 0$:

Theorem ([**Sch83**])**.** *Let Σ be a properly embedded minimal annulus in $\mathbb{R}^3$ such that the total curvature $\int_\Sigma \kappa_1 \cdot \kappa_2 \, dA$ is finite. Then Σ is a catenoid.*

Recall that a subset S of a topological space M is said to be *proper* if $S \cap K$ is compact in M for all compact subsets K of M. A map $f \colon \Omega \to M$ of a topological space Ω into M is *proper* if $f^{-1}(K)$ is compact in Ω for all compact $K \subset M$.

We will not explain the other technical assumptions. This theorem indicates that the classification of MCH-annuli is more complicated than the classification of MCH-spheres. Schoen used Alexandrov's moving planes argument for the proof. Some years later, based on results by Meeks, Korevaar, Kusner (a student of Rick Schoen) and Solomon solved the problem for MCH-annuli in $\mathbb{R}^3$:

Theorem ([**KKS89**])**.** *Let Σ be a properly embedded* MCH*-annulus in $\mathbb{R}^3$. Then Σ is a Delaunay unduloid.*

The proof, once again, employs a version of Alexandrov reflection.

Constant mean curvature surfaces in homogeneous manifolds

Surfaces of constant mean curvature have been also studied in various Riemannian 3-manifolds. In the 1980s, Thurston conjectured that a Riemannian 3-manifold can be decomposed into canonical 3-dimensional geometries. Each of these canonical 3-dimensional geometries is a simply connected homogeneous 3-manifold. Homogeneity is an important notion so that we define it here:

Definition. A Riemannian manifold M is said to be *homogeneous* if for all $p, q \in M$ there is an isometry φ of M such that $\varphi(p) = q$. A homogeneous manifold M is *isotropic* if for all $p, q \in M$, all unit vectors $v \in T_pM$ and all unit vectors $w \in T_qM$ there is an isometry φ of M such that

$$\varphi(p) = q \quad \text{and} \quad d\varphi_p v = w.$$

Hence, in a homogeneous 3-manifold the geometry does not depend on the point, and in an isotropic manifold the geometry is also independent of the direction.

The isometry group $\mathrm{Iso}(M)$ of a simply connected homogeneous 3-manifold M has dimension 3, 4 or 6. Let $\mathbb{H}^n$, $\mathbb{R}^n$ and $\mathbb{S}^n$ denote the simply connected n-manifolds of constant curvature -1, 0 and 1, respectively. Then the canonical 3-dimensional geometries are realised by the following models:

Model geometry	$\dim(\mathrm{Iso}(M))$	Geometric structure	Riemannian fibration over
$\mathbb{H}^3$	6	isotropic	–
$\mathbb{R}^3$	6	isotropic	$\mathbb{R}^2$
$\mathbb{S}^3$	6	isotropic	$\mathbb{S}^2$
$\mathbb{H}^2 \times \mathbb{R}$	4	non-isotropic	$\mathbb{H}^2$
$\mathbb{S}^2 \times \mathbb{R}$	4	non-isotropic	$\mathbb{S}^2$
$\widetilde{\mathrm{PSL}}(2, \mathbb{R})$	4	non-isotropic	$\mathbb{H}^2$
Nil_3	4	non-isotropic	$\mathbb{R}^2$
Sol_3	3	non-isotropic	$\mathbb{R}$

Similar results to the ones mentioned for $\mathbb{R}^3$ have been established in $\mathbb{H}^3$. For instance, Korevaar, Kusner, Meeks and Solomon showed in [**KKMS92**] that a properly embedded MCH-annulus in $\mathbb{H}^3$ with $H > 1$ is rotationally invariant. However, an extensive study of MCH-surfaces in the other model geometries started only recently after Perelman proved Thurston's conjecture in 2003. In 2004, Abresch and Rosenberg generalised the classical Hopf theorem from $\mathbb{R}^3$ to all simply connected 3-dimensional homogeneous manifolds with a 4-dimensional isometry group: immersed MCH-spheres are rotationally invariant. Hopf's theorem relies on the existence of a holomorphic quadratic differential, of which Abresch and Rosenberg found a generalisation in [**AR05**]. In Sol_3, a homogeneous space with a 3-dimensional isometry, existence and uniqueness of MCH-spheres is more complicated and was established with different methods by Daniel, Mira and Meeks in [**DM13**] and [**Mee13**], respectively.

Since MCH-spheres in homogeneous 3-manifolds are well-understood, it is natural to consider MCH-annuli in these ambient spaces, for example in the product $\mathbb{H}^2 \times \mathbb{R}$. With respect to the Riemannian fibration $\mathbb{H}^2 \times \mathbb{R} \to \mathbb{H}^2$, $(p, h) \mapsto p$ there are three types of geodesics: vertical, horizontal and tilted geodesics. Arbitrary rotations about vertical geodesics and rotations of angle π about horizontal geodesics are isometries, but we cannot rotate at all about tilted geodesics. Nevertheless, we have isometric translations along each geodesic.

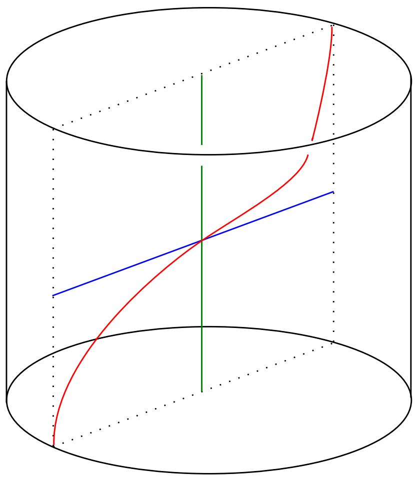


FIGURE E. A model of $\mathbb{H}^2 \times \mathbb{R}$ where $\mathbb{H}^2$ is represented by the Poincaré disc model. The green, blue and red curves represent a vertical, horizontal and tilted geodesic, respectively.

Keeping in mind that a Delaunay unduloid in $\mathbb{R}^3$ is singly periodic with respect to translations along the axis of rotation, the following conjecture due to Meeks is the natural generalisation of [**KKS89**] in $\mathbb{H}^2 \times \mathbb{R}$:

Conjecture (Meeks, 2010)**.** *Let Σ be a properly embedded* MCH*-annulus in $\mathbb{H}^2 \times \mathbb{R}$ with $H > \frac{1}{2}$. Then Σ stays at a bounded distance from a geodesic of $\mathbb{H}^2 \times \mathbb{R}$ and is singly periodic with respect to translations along that geodesic.*

Content and structure of this thesis

Meeks' conjecture is the starting point of the present thesis. We consider the following existence problem underlying the conjecture:

Conjecture. *Let $H > \frac{1}{2}$ and let γ be a geodesic in $\mathbb{H}^2 \times \mathbb{R}$ with slope $\alpha \in [0, \pi/2]$ with respect to a vertical geodesic. Then there exists a one-parameter family $(\Sigma^{c,\alpha})_{c\in(0,1)}$ of properly embedded* MCH*-annuli which are singly periodic with respect to translations along γ; we refer to them as* unduloids with axis γ*. If γ is tilted we call the surfaces* tilted unduloids.

The parameter c corresponds to the neck size *of $\Sigma^{c,\alpha}$, that is, c parametrises the length of the shortest closed geodesic on $\Sigma^{c,\alpha}$. The limiting cases are: $\Sigma^{0,\alpha}$, a chain of* MCH*-spheres aligned along γ, and $\Sigma^{1,\alpha}$, a cylinder with tilted axis.*

See Figure F for a qualitative sketch of these surfaces. There are known examples of singly periodic MCH-annuli when the geodesic is vertical (see [**HH89**]) or horizontal (see [**MT14**]) but examples with a tilted axis have not been established. The examples with a vertical geodesic constructed by Hsiang in [**HH89**] are rotationally invariant surfaces and generalise the Delaunay surfaces from $\mathbb{R}^3$; these *vertical unduloids* are solutions of an ODE. The surfaces constructed by Manzano and Torralbo in [**MT14**] can be considered as *horizontal unduloids* and they are proper PDE solutions. A conjugate Plateau construction à la Große-Brauckmann in [**GB93**] has been developed to construct one quarter of the desired surface and extend it by reflections. This ansatz cannot be used directly for a tilted geodesic.

To answer this existence problem with respect to a tilted geodesic we distinguish two classes of problems:

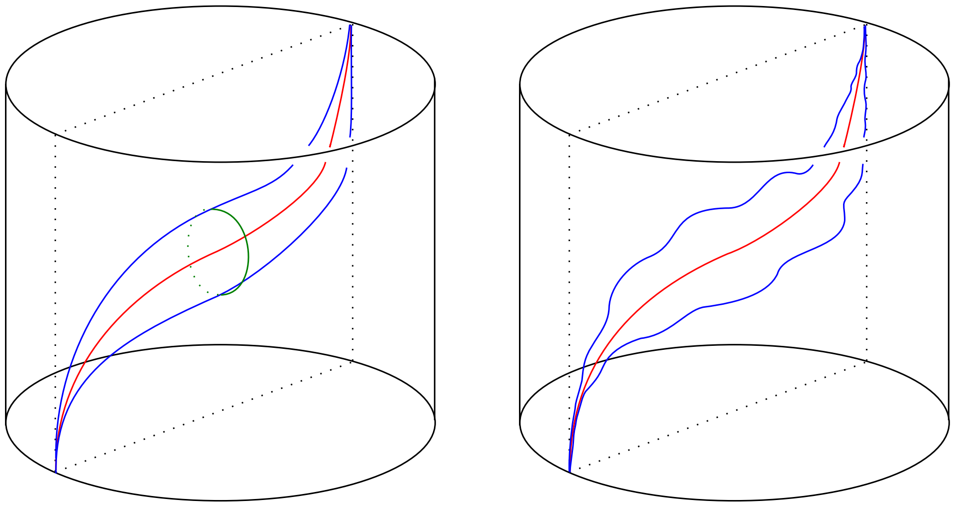


FIGURE F. Left: Qualitative sketch of a tilted cylinder in $\mathbb{H}^2 \times \mathbb{R}$. The cylinder is invariant under translations along the geodesic γ shown in red. The simple closed curve shown in green generates the cylinder. The dotted vertical plane containing γ intersects the cylinder in the geodesics shown in blue.
Right: Similar the intersection pattern for a more general singly periodic surface with the dotted plane. The blue curves are periodic.

***Cylinder case*:** translationally invariant surfaces arise as solutions of an ODE.

***Unduloid case*:** MCH-surfaces invariant under a discrete group of translations are solutions of a PDE (due to lack of rotations about a non-vertical axis).

We address these problems in the two parts of the present thesis.

Content of Part 1: Translationally invariant MCH-cylinders

In the first part we study invariant surfaces with constant mean curvature. When the ODE these surfaces satisfy has a simple closed curve as solution we call the invariant surface generated by this curve an MCH-*cylinder*. Our approach to this problem is as follows:

We consider invariant MCH-surfaces generated by graphical curves in planes. A comparison with MCH-spheres yields properties of the graph which allows us to extend the graph to a simple closed solution curve (as illustrated in Figure F by the green curve on page xvi). This approach is sufficient to obtain tilted MCH-annuli in $\mathbb{H}^2 \times \mathbb{R}$. As a by-product we obtain also first examples of MCH-annuli in various homogeneous 3-manifolds, for example in Sol_3 and in $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$. This part of the present thesis is available as a preprint on the ArXiv, see [**Vrz14**].

The ideas for this part were initially developed to show existence of tilted MCH-cylinders in $\mathbb{H}^2 \times \mathbb{R}$. Only later we considered the less symmetric manifold Sol_3. It is, however, instructive to present the work in Sol_3, as we chose to do in the first chapter of this part, in order to understand which geometric properties and structures are needed to prove existence of MCH-cylinders.

At each point of Sol_3, which we consider as a Riemannian fibration $\mathrm{Sol}_3 \to \mathbb{R}$ with $\mathbb{R}^2$-fibres and base $\mathbb{R}$, there are three distinguished geodesics which admit rotations of angle π: the base c_1 and two orthogonal lines c_2, c_3 in the $\mathbb{R}^2$-fibre. Since Sol_3 is also a metric Lie group, left-translations along any of these three geodesics define a one-parameter family of isometries. For each of these geodesics we construct surfaces which are invariant under the corresponding family of left-translations. Without the need to state the ODE explicitly we prove existence of embedded MCH-cylinders in Sol_3 with $H > 0$. The following main result summarises Theorem 1.5 on page 13 and Theorem 1.11 on page 20:

Main Result 1. *For each $H > 0$ and each $j \in \{1,2,3\}$ there is a smooth embedded simple closed curve γ_j which generates an embedded MCH-cylinder invariant under translations along c_j.*

The MCH-cylinders invariant under translations along c_1 have been conjectured to exist by Lopez in [**Lop14**], on the grounds of computed examples. We note that surfaces invariant under translations along c_2 and c_3 have not yet been considered, and so our families should be the first examples of embedded annuli with constant mean curvature $H > 0$ in Sol_3.

We also include images of further computed examples of MCH-cylinders in Sol_3 which have the same invariance, but are only immersed. See the title figures as well as Figure 1.8 and Figure 1.9 on pages 18 and 19, respectively. This class seems rich: We conjecture there are infinitely many simple closed solution curves with self-intersections which generate MCH-cylinders. To be more detailed, we summarise it as follows:

Experimental Result 2. *For $H > 0$ there are non-embedded curves γ which generate* MCH*-cylinders in* Sol_3. *For $H = 1$, the curve γ can have turning number $5 + 4k$, where $k \in \{1, 2, 3, 4\}$. There is $H_0 \in (0, 1)$ such that for $H \in (0, H_0)$ non-embedded solution curves with turning number 5 exist and converge to a multiple cover of the embedded* MCH*-cylinder as $H \to H_0$.*

In a second chapter we consider Riemannian fibrations $E \to B$ with geodesic fibres. They are parametrised as $E(\kappa, \tau)$-spaces with base curvature κ and bundle curvature τ. The $E(\kappa, \tau)$-spaces have 4- or 6-dimensional isometry groups. In case of a 4-dimensional isometry groups, rotations about non-vertical geodesics need not be isometries, and their respective geodesic tubes need not have constant mean curvature. However, translations along geodesics are still isometries. We exclude the compact case of the Berger spheres, which admit MCH-spheres which are possibly self-intersecting; comparison spheres for our geometric approach are not available.

A reasoning similar to Chapter 1 yields existence of MCH-cylinders with $H > H(E)$, invariant under translation along those geodesic axes which have a geodesic projection into the base space B, see Theorem 2.7. For $\tau = 0$ any geodesic has a geodesic projection, for $\tau \neq 0$ only horizontal (and vertical) geodesics have a geodesic projection. For $\kappa = -1$ and $\tau = 0$ this theorem includes *tilted* MCH-cylinders in $\mathbb{H}^2 \times \mathbb{R}$; we also get horizontal MCH-cylinders in $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$. Again, we do not need to refer to the explicit form of the ODE. In Theorem 2.10 we calculate the horizontal diameter of these surfaces. The argument is based on a weight formula for MCH-surfaces. We summarise these theorems:

Main Result 3. *Let c be a geodesic in a non-compact $E(\kappa, \tau)$-space with geodesic projection and let Γ be the family of translations along c. For each $H > H(E)$ there is a smoothly embedded simple closed curve β which generates a Γ-invariant cylinder f with constant mean curvature H. The surface f is embedded, except in $\mathbb{S}^2 \times \mathbb{R}$.*

The horizontal diameter of a horizontal MCH*-cylinder in a non-compact $E(\kappa, \tau)$-space with $\kappa \leq 0$ is*

$$\frac{2}{\sqrt{-\kappa}} \operatorname{arctanh}\left(\frac{\sqrt{-\kappa}}{2H}\right).$$

Content of Part 2: Singly periodic MCH-annuli in $\mathbb{H}^2 \times \mathbb{R}$

In the second part we study the more general existence problem for singly periodic MCH-annuli in $\mathbb{H}^2 \times \mathbb{R}$. A specific difficulty is to propose a clear-cut problem whose solution results in the desired surfaces. A known method for the construction of periodic MCH-surfaces with *symmetries* is the *conjugate Plateau construction*. It is based on the *Daniel correspondence*: If an MCH-surface with $H > \frac{1}{2}$ in $\mathbb{H}^2 \times \mathbb{R}$ has a mirror plane then there exists an isometric minimal surface in a compact $E(\kappa, \tau)$-space, a so-called *Berger sphere*, and the minimal surface is bounded by certain geodesics; vice versa, a minimal surface bounded by certain certain geodesics in this Berger sphere corresponds to an MCH-surface in $\mathbb{H}^2 \times \mathbb{R}$ with $H > \frac{1}{2}$ and the surface has symmetry planes. Manzano and Torralbo used this method to construct a horizontal unduloid in $\mathbb{H}^2 \times \mathbb{R}$.

In order to apply a conjugate Plateau construction, we study symmetries of singly periodic (Alexandrov) embedded MCH-annuli in $\mathbb{H}^2 \times \mathbb{R}$. We see that they always have a vertical mirror plane, separating the surface into two halves. Each half is an MCH-surface in $\mathbb{H}^2 \times \mathbb{R}$ bounded by two curves. Revisiting the known singly periodic examples in $\mathbb{H}^2 \times \mathbb{R}$, we see that these symmetry curves in the vertical mirror plane correspond to linked and closed geodesics in a Berger sphere, and the corresponding minimal surface is a minimal annulus bounded by these two linked curves; see Figure G. We prove that any pair of linked horizontal geodesics in a Berger sphere bounds at least two minimal annuli, a “Min“ and a “Minimax“. If these are the only minimal annuli bounded by a given pair of linked horizontal geodesics we can show that tilted unduloids in $\mathbb{H}^2 \times \mathbb{R}$ exist. Thus the existence problem for tilted unduloids in $\mathbb{H}^2 \times \mathbb{R}$ is reduced to a uniqueness problem for minimal annuli in the Berger spheres. It remains to settle the uniqueness problem.

In Chapter 3 we introduce all the technical tools for the conjugate Plateau construction and we define singly periodic surfaces. Using Alexandrov’s reflections principle we show that singly periodic (Alexandrov) embedded MCH-annuli with $H > \frac{1}{2}$ in $\mathbb{H}^2 \times \mathbb{R}$ have a vertical mirror plane and thus decompose into two simply connected pieces with boundary in a vertical plane, that is, each piece solves a free boundary problem. By Daniel’s correspondence, this MCH-surface has a minimal sister surface in the compact space $E(4H^2-1, H)$, and this minimal surface is bounded by horizontal geodesics: a free boundary problem in $\mathbb{H}^2 \times \mathbb{R} = E(-1, 0)$ is thus reduced to a fixed boundary problem in $E(4H^2 - 1, H)$. However, the geometry of these compact homogeneous 3-manifolds

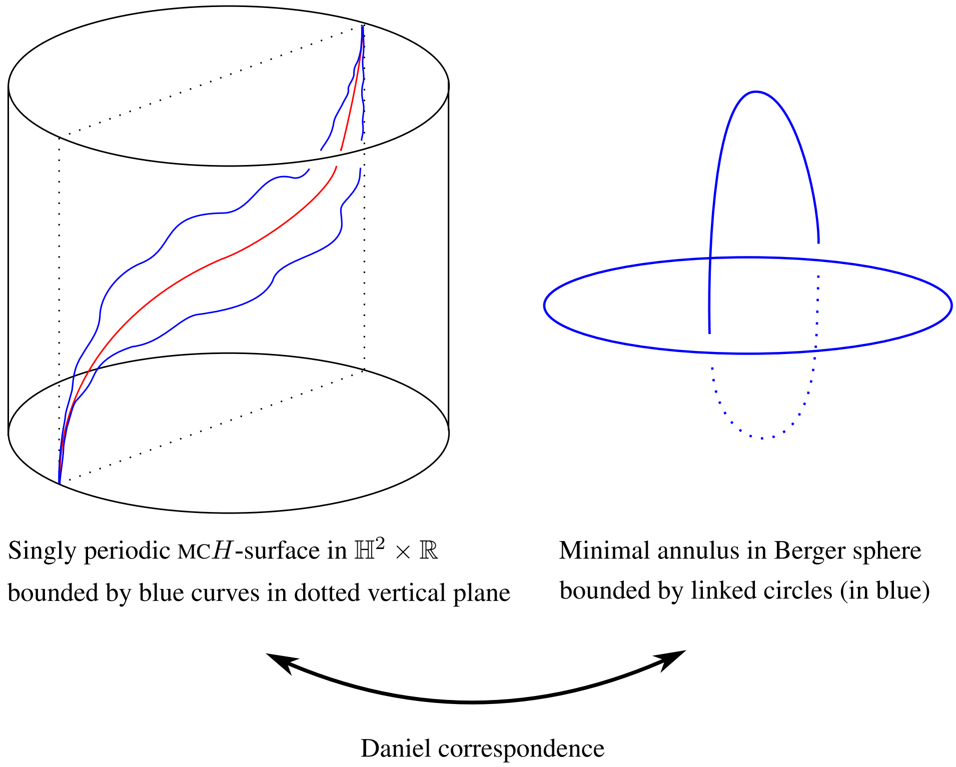


FIGURE G. Illustration of Daniel correspondence

with 4-dimensional isometry group is more complicated, and is therefore studied in detail in this chapter.

We study the geometry of the known solutions in the subsequent chapter in order to determine the correct boundary contour as well as the topology of the desired minimal surface in order to apply the conjugate Plateau construction. It turns out that suitable horizontal geodesics are linked and bound minimal annuli. Moreover, we show the following result:

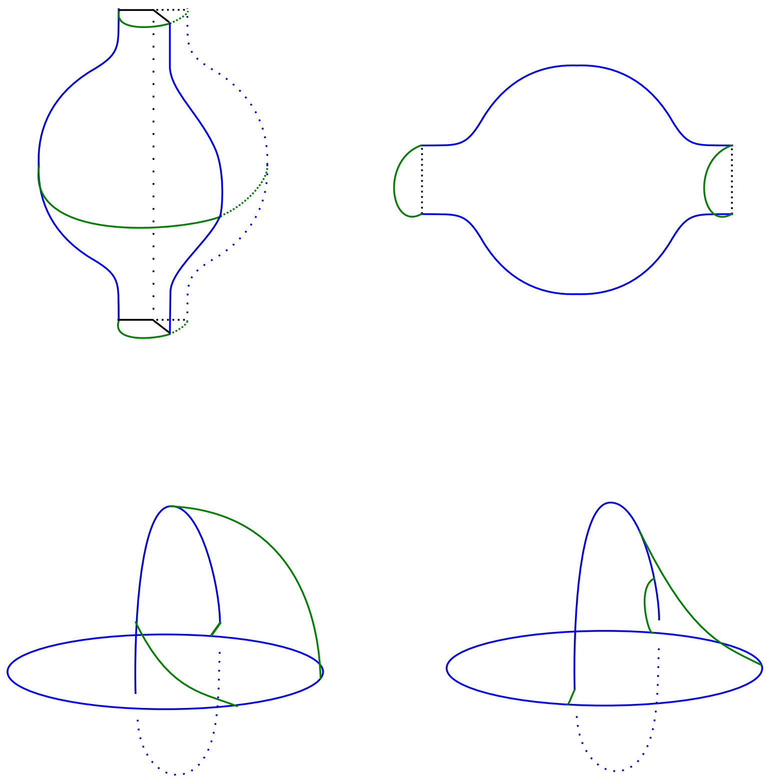


FIGURE H. Illustration of Main Result 4: On the left side one minimal annulus and the corresponding piece of a vertical unduloid in $\mathbb{H}^2 \times \mathbb{R}$, the dotted blue line in $\mathbb{H}^2 \times \mathbb{R}$ indicating one half of a vertical unduloid. Right: second minimal annulus corresponding to one half of a horizontal unduloid in $\mathbb{H}^2 \times \mathbb{R}$.

Main Result 4. *Let h_1 and h_2 be linked horizontal geodesics in the Berger sphere $E(4H^2-1, H)$ that are integral curves of the same horizontal field F and which are not too far away from each other. Then h_1 and h_2 bound at least two minimal annuli.*

Under the Daniel correspondence, one minimal annulus relates to one half of a horizontal unduloid in $\mathbb{H}^2 \times \mathbb{R}$, while the other minimal annulus relates to a piece of a vertical unduloid in $\mathbb{H}^2 \times \mathbb{R}$.

This result naturally opens up the question whether a multiple solution theorem holds for any pair of linked horizontal geodesics in a Berger sphere. Indeed, in Chapter 5 we prove the following multiple solution theorem in the compact $E(\kappa, \tau)$-spaces, see also Theorem 5.5:

Main Result 5. *Let h_1 and h_2 be linked horizontal geodesics in a Berger sphere. Then there are at least two minimal annuli bounded by h_1 and h_2 (for a prescribed orientation of h_1 and h_2).*

One of the solutions is well-known, it is a *spherical helicoid*. The spherical helicoid in the Berger sphere $E(4H^2 - 1, H)$ corresponds to a piece of a vertical unduloid, but we obtain also many new minimal annuli in the Berger spheres. The multiple solution theorem is an application of a more general minimax principle in compact Riemannian 3-manifolds that was proved by Ji Min [**Min93a**]. In order to apply this principle we provide the setup from [**Min93a**] in Section 5.2. In Appendix B we provide the details from [**Min93a**], in conjunction with the results of related papers.

In our final chapter we reduce the existence problem for singly periodic (Alexandrov) embedded MCH-annuli in $\mathbb{H}^2 \times \mathbb{R}$ to a uniqueness problem for minimal annuli in the Berger spheres. We apply the conjugate Plateau construction as follows: We start with a minimal annulus bounded by linked horizontal geodesics h_1 and h_2 in the Berger sphere $E(4H^2 - 1, H)$. We define hypotheses (H1) to (H3) for such a minimal annulus. If a minimal annulus satisfies (H1) to (H3) we show in Theorem 6.1 that the MCH-sister surface in $\mathbb{H}^2 \times \mathbb{R}$ extends to a tilted unduloid. In Section 6.2 we discuss how these hypotheses can be verified under the assumption that there are exactly two minimal annuli bounded by h_1 and h_2. This yields the following main result:

Main Result 6. *If for any pair of linked horizontal geodesics h_1 and h_2 there are exactly two minimal annuli bounded by h_1 and h_2 (for a prescribed orientation of the boundary) then there exist tilted unduloids in $\mathbb{H}^2 \times \mathbb{R}$.*

In the Outlook we briefly cover further topics, including open problems, conjectures or approaches related to results in this thesis. These ideas can serve for future projects building upon this thesis.

Part 1

Translationally invariant ${}_{\text{MC}}H$-cylinders in homogeneous 3-manifolds

CHAPTER 1

MCH-cylinders in Sol_3

In this chapter we study invariant surfaces in the homogeneous 3-manifold Sol_3 with a 3-dimensional isometry group. This lack of symmetries makes it more difficult to construct examples of MCH-surfaces. Some new ideas are needed to establish existence of MCH-cylinders in Sol_3, that is, an MCH-surface generated by a simple closed curve under a group of translations.

In Section 1.1 of this chapter we describe the metric Lie group Sol_3 as a semi-direct product $\mathbb{R}^2 \ltimes_A \mathbb{R}$. We note that Sol_3 can be considered a Riemannian fibration via the projection $\mathrm{Sol}_3 \to \mathbb{R}$, $(x, y, z) \mapsto z$. At each point of Sol_3 there are three orthogonal geodesics with the following properties: one is the base and the other two are orthogonal geodesics in a $\mathbb{R}^2$-fibre, these three geodesics admit geodesic reflections and left-translations along them are isometries. These geometric properties and the existence of embedded MCH-spheres are the main ingredients to prove existence of MCH-cylinders in Sol_3.

Section 1.2 is devoted to constant mean curvature surfaces invariant under translations along the base. One problem concerns the ODE satisfied by a graph generating such a surface. The other problem is the geometric discussion of the ODE and the extension of the graphical solution to a simple closed embedded curve. Classical approaches to this problem are based either on explicit solutions or qualitative discussions involving first integrals. For $H = 0$ these techniques yield a partial classification of invariant minimal surfaces, see [**Lop14**, Section 2], but not so for $H > 0$. We solve this problem by comparing graphical solutions with MCH-spheres in Sol_3 to obtain qualitative properties. The main result is stated in Theorem 1.5.

For this class of surfaces we also include images of computed examples. This leads to conjectures on the embedded MCH-cylinders varying in H. Moreover, we find interesting non-embedded simple closed curves as generators of MCH-cylinders.

In Section 1.3 we proceed analogously and construct MCH-cylinders invariant under translations along geodesics admitting geodesic reflections in a $\mathbb{R}^2$-fibre of Sol_3 and the result is stated in Theorem 1.11. These invariant surfaces have not been considered before. There are several other geodesics in a $\mathbb{R}^2$-fibre of Sol_3 which admit no isometric rotations at all. These are usually considered for invariant surfaces in Sol_3; see for instance [**LM14**] or [**DM13**, Example 3.9].

1.1. Preliminaries on Sol_3

The space Sol_3 is a simply connected homogeneous 3-manifold diffeomorphic to $\mathbb{R}^3$ and as such a metric Lie group. We describe a model for this space and some properties.

Model. We endow $\mathbb{R}^3$ with the Riemannian metric

$$\langle \cdot, \cdot \rangle_{(x,y,z)} = e^{2z} dx^2 + e^{-2z} dy^2 + dz^2, \tag{1.1}$$

and set $\mathrm{Sol}_3 := \big(\mathbb{R}^3, \langle \cdot, \cdot \rangle\big)$. The multiplication

$$(x_1, y_1, z_1) * (x_2, y_2, z_2) := \big(x_1 + e^{-z_1} x_2, y_1 + e^{z_1} y_2, z_1 + z_2\big) \tag{1.2}$$

turns Sol_3 into a metric Lie group, i.e., for $a \in \mathrm{Sol}_3$ the left-multiplication

$$\mathcal{L}_a \colon \mathrm{Sol}_3 \to \mathrm{Sol}_3, \qquad \mathcal{L}_a(g) := a * g$$

is an isometry of Sol_3.

We remark that Sol_3 can be considered a Riemannian fibration via the projection $\mathrm{Sol}_3 \to \mathbb{R}$, $(x, y, z) \mapsto z$ with $\mathbb{R}^2$-fibres over the z-axis.

Canonical frame and Riemannian connection. At the origin let $(\partial_x, \partial_y, \partial_z)$ be the standard Euclidean frame. A left-translation from the origin to $p = (x, y, z)$ gives the orthonormal frame

$$E_1 = e^{-z} \partial_x, \qquad E_2 = e^{z} \partial_y, \qquad E_3 = \partial_z.$$

The Riemannian connection with respect to this frame has the following representation:

$$\begin{array}{lll} \nabla_{E_1} E_1 = -E_3, & \nabla_{E_1} E_2 = 0, & \nabla_{E_1} E_3 = E_1, \\ \nabla_{E_2} E_1 = 0, & \nabla_{E_2} E_2 = E_3, & \nabla_{E_2} E_3 = -E_2, \\ \nabla_{E_3} E_1 = 0, & \nabla_{E_3} E_2 = 0, & \nabla_{E_3} E_3 = 0. \end{array}$$

Special geodesics and induced isometries. We consider the unit-speed geodesics

$$c\colon \mathbb{R} \to \mathrm{Sol}_3, \quad c(s) := (0,0,s) \qquad \text{and} \qquad c_\pm\colon \mathbb{R} \to \mathrm{Sol}_3, \quad c_\pm(s) := \frac{1}{\sqrt{2}}(s, \pm s, 0).$$

Since Sol_3 is a metric Lie group we obtain a one-parameter family of isometries $(\Phi_s)_{s\in\mathbb{R}}$ by setting

$$\Phi_s\colon\ \mathrm{Sol}_3 \to \mathrm{Sol}_3, \qquad \Phi_s(x,y,z) := \mathcal{L}_{c(s)}(x,y,z) = (e^{-s}x, e^{s}y, z+s), \qquad s \in \mathbb{R}.$$

We call the family $\Gamma := (\Phi_s)_{s\in\mathbb{R}}$ *translations along* c.

Another one-parameter family of isometries $\Gamma_\pm := (\Phi_{\pm,s})_{s\in\mathbb{R}}$, *translations along* $c_\pm$ *in* Sol_3, is defined by

$$\Phi_{\pm,s}\colon\ \mathrm{Sol}_3 \to \mathrm{Sol}_3,\ (x,y,z) \mapsto \mathcal{L}_{c_\pm(s)}(x,y,z) = \left(x + \frac{s}{\sqrt{2}}, y \pm \frac{s}{\sqrt{2}}, z\right),\ s \in \mathbb{R}.$$

Each x-z plane is a totally geodesic submanifold of Sol_3: Indeed, E_2 is normal and $\langle \nabla_{E_i} E_j, E_2\rangle = 0$ for $i,j \in \{1,3\}$. Similarly the y-z planes are totally geodesic, too, with normal E_1 and $\langle \nabla_{E_i} E_j, E_1\rangle = 0$ for $i,j \in \{2,3\}$. In particular, reflections in all these planes are isometries of Sol_3. We denote the reflection in $\{x = 0\}$ by σ_{yz} and reflection in $\{y = 0\}$ by σ_{xz}.

The following two properties are less obvious and can be verified directly from the form of the metric (1.1): Rotations by an angle π about c and $c_\pm$ are isometries of Sol_3. We denote them by

$$\rho\colon\ \mathrm{Sol}_3 \to \mathrm{Sol}_3, \quad \rho(x,y,z) := (-x,-y,z)$$

and

$$\rho_\pm\colon\ \mathrm{Sol}_3 \to \mathrm{Sol}_3, \quad \rho_\pm(x,y,z) := (\pm y, \pm x, -z).$$

Killing fields. Left-translation along each of the coordinate axes defines a one-parameter group of isometries, generated by the following three Killing fields:

$$K_1 = e^{z}E_1 = \partial_x, \qquad K_2 = e^{-z}E_2 = \partial_y, \qquad K_3 = -xK_1 + yK_2 + E_3.$$

This is useful to describe Killing graphs defined on $\{y = 0\}$ as well as for translations in the x-y plane $\{z = 0\}$. We will use this later.

Constant mean curvature spheres. In [**Mee13**] MCH-spheres in Sol_3 are studied. We need the following property of MCH-spheres in Sol_3:

Proposition 1.1. *Let $H > 0$ and $\mathbb{S}_H$ be a sphere of constant mean curvature H in* Sol_3*, centred at* $(0,0,0)$*. Then $\{x = 0\}$ and $\{y = 0\}$ are mirror planes of $\mathbb{S}_H$ and $\mathbb{S}_H$ is a bi-graph with respect to each mirror plane. The minimal and maximal values of the x, y and z coordinates arise on the respective coordinate axes.*

Proof. The first part is stated in [**Mee13**]. The last claim is a consequence of the Gauß map being a diffeomorphism: If minimum and maximum were attained elsewhere the Gauß map could not be injective because $\mathbb{S}_H$ is invariant by rotations of angle π about the z-axis and $\mathbb{S}_H$ is invariant under reflection through $\{x = 0\}$ and $\{y = 0\}$. □

1.2. Surfaces invariant under translations along c

In this section we study constant mean curvature surfaces invariant under translation along the base c of Sol_3. First we describe properties of the differential equation for constant mean curvature surfaces invariant by Γ. These are natural implications by the geometry of Sol_3. Then we discuss the solution of this ODE geometrically. We use the maximum principle to derive properties which let us extend the respective graph by reflections to an embedded closed solution curve. We also discuss further solutions which we obtained computationally.

1.2.1. ODE for surfaces invariant under translations along c. The foliation by x-y planes of Sol_3 stays invariant under translations along c. Therefore it is sufficient to consider a curve in the fibre

$$S_0 := \{(x, y, z) \in \mathrm{Sol}_3 \colon z = 0\}$$

as generating curve of a surface invariant by translation along c.

Explicitly, for $\mathcal{C}^2$-functions $x \colon J \to \mathbb{R}$ and $y \colon J \to \mathbb{R}$, defined on an open interval $J \subset \mathbb{R}$, the curve

$$\gamma \colon J \to \mathrm{Sol}_3, \qquad \gamma(t) := (x(t), y(t), 0)$$

is in S_0 and the invariant surface generated by translation of γ along c is parametrised by

$$f \colon \mathbb{R} \times J \to \mathrm{Sol}_3, \qquad f(s,t) := \Phi_s(\gamma(t)) = \big(e^{-s}x(t), e^{s}y(t), s\big)\,. \tag{1.3}$$

The mean curvature of f is independent of s, i.e., we have $H = H(t)$. Requiring H to be constant leads to an ordinary differential equation for γ. Such surfaces were studied in [**Lop14**], too. For $H = 0$ some initial value problems have explicit solutions or allow for qualitative discussions involving first integrals. For $H > 0$, however, the mean curvature equation appears too complicated for these approaches.

We will consider graphical solutions, for which the ODE can be described as follows:

Proposition 1.2. *Let $H \in \mathbb{R}$.*

(a) There is a smooth function $F\colon \mathbb{R}^3 \to \mathbb{R}$ such that the invariant surface

$$f\colon \mathbb{R} \times J \to \mathrm{Sol}_3, \qquad f(s,t) := \Phi_s(t, h(t), 0), \qquad \textit{where } h \in \mathcal{C}^2(J, \mathbb{R}),$$

has constant mean curvature H with respect to the upper normal if and only if

$$h''(t) = F(t, h(t), h'(t)) \qquad \textit{for all } t \in J. \tag{1.4}$$

(b) The invariant surface $\tilde{f}\colon \mathbb{R} \times J \to \mathrm{Sol}_3$, $\tilde{f}(s,t) := \Phi_{-s}(h(t), t, 0)$ has constant mean curvature H if and only if $h \in \mathcal{C}^2(J, \mathbb{R})$ satisfies (1.4)*, i.e., x-graphs and y-graphs as generating curves of invariant surfaces with constant mean curvature H satisfy the same ODE.*

Proof. (a) Let $v_1 := \partial_s f$ and $v_2 := \partial_t f$. We denote the upper normal to f by N, so that $g_{ij} := \langle v_i, v_j \rangle$ and $b_{ij} := \langle \nabla_{v_i} v_j, N \rangle$ for $i, j \in \{1, 2\}$ are the coefficients of the first and second fundamental form. Then the mean curvature of f is given by

$$H = \frac{b_{11} g^{11} + 2 b_{12} g^{12} + b_{22} g^{22}}{2}.$$

We have

$$v_2 = E_1 + h' E_2 \qquad \text{and} \qquad \nabla_{v_2} v_2 = \underbrace{\nabla_{v_2} E_1 + h' \nabla_{v_2} E_2}_{=:w} + h'' E_2$$

Here we note that H depends on t, $h(t)$, $h'(t)$ and $h''(t)$.

We assume H to be constant and therefore get an implicit differential equation depending on h' and h''. Now we want to show that we can solve this implicit equation for h''.

Obviously w is independent of h'' and the only term containing h'' is

$$\frac{b_{22} g^{22}}{2} = \frac{b_{22} g_{11}}{2 \det(g)} = \frac{\langle \nabla_{v_2} v_2, N \rangle g_{11}}{2 \det(g)} = \frac{\left(h'' \langle N, E_2 \rangle + \langle w, N \rangle \right) g_{11}}{2 \det(g)}.$$

The surface f is a Killing graph with respect to the Killing field $K_2 = \partial y = e^{-s}E_2$, so that $\langle N, E_2\rangle$ is positive, because N is chosen as upper normal. We also have $g_{11} > 0$ because the Killing field generated by translation along c is non-trivial. Therefore we can solve the implicit equation for h'' and get a function $F\colon \mathbb{R}^3 \to \mathbb{R}$ with $h''(t) = F(t, h(t), h'(t))$. The function F is smooth because each Φ_s is smooth and so are g and b. It is defined on all of $\mathbb{R}^3$ because we can prescribe any kind of function $h\colon J \to \mathbb{R}$.

(b) The equation $\Phi_{-s} \circ \rho_+ = \rho_+ \circ \Phi_s$ implies $\tilde{f} = \rho_+ \circ f$, i.e. $\tilde{f}$ and f are isometric. Thus the claim about the ODE follows from (a). $\square$

1.2.2. Half-cylinder solution and its extension to MCH-cylinders with axis c. We consider the ODE for surfaces invariant by translations along the base c first. We can apply the Picard-Lindelöf Theorem to (1.4) because F is smooth. We obtain a maximal solution h. For constant mean curvature $H > 0$ the maximum principle yields some properties by comparing the surface f with spheres of constant mean curvature H, which will justify the name “half-cylinder”:

Lemma 1.3 (Half-cylinder solution)**.** *Given $a, b \in \mathbb{R}$ and $H > 0$, there is a unique maximal solution $h\colon I_{\max} \to \mathbb{R}$ with $h(0) = a$ and $h'(0) = b$ satisfying* (1.4). *For each $a, b \in \mathbb{R}$ it has the following properties:*

(a) [x-boundedness]: *There are real numbers $R_- = R_-(a,b) < R_+ = R_+(a,b)$ such that $I_{\max} = (R_-, R_+)$.*

(b) [y-boundedness]: *There is $K = K(a,b) > 0$ such that* $\lim_{t\to R_\pm} |h(t)| \le K$.

(c) [Asymptotic behaviour]: *We have* $\lim_{t\to R_\pm} h'(t) = \pm\infty$.

(d) [Monotonicity]: *There is $t_0 \in (R_-, R_+)$ with $h'(t_0) = 0$. On (R_-, t_0) the function h is monotonically decreasing and on (t_0, R_+) it is monotonically increasing.*

(e) [Symmetry]: *For $b = 0$ we have $R = R(a) := R_+(a, 0) = -R_-(a, 0)$ for the maximal solution with $h(0) = a$ and $h'(0) = 0$.*

Proof. Let $h\colon I_{\max} \to \mathbb{R}$ be the unique maximal solution of $h''(t) = F(t, h(t), h'(t))$ with $h(0) = a$ and $h'(0) = b$ and Σ be the surface generated by $(t, h(t), 0)$. We will use frequently that translations along the x-axis or y-axis are isometries of Sol_3 and that f is an invariant surface. Here we also use Proposition 1.1.

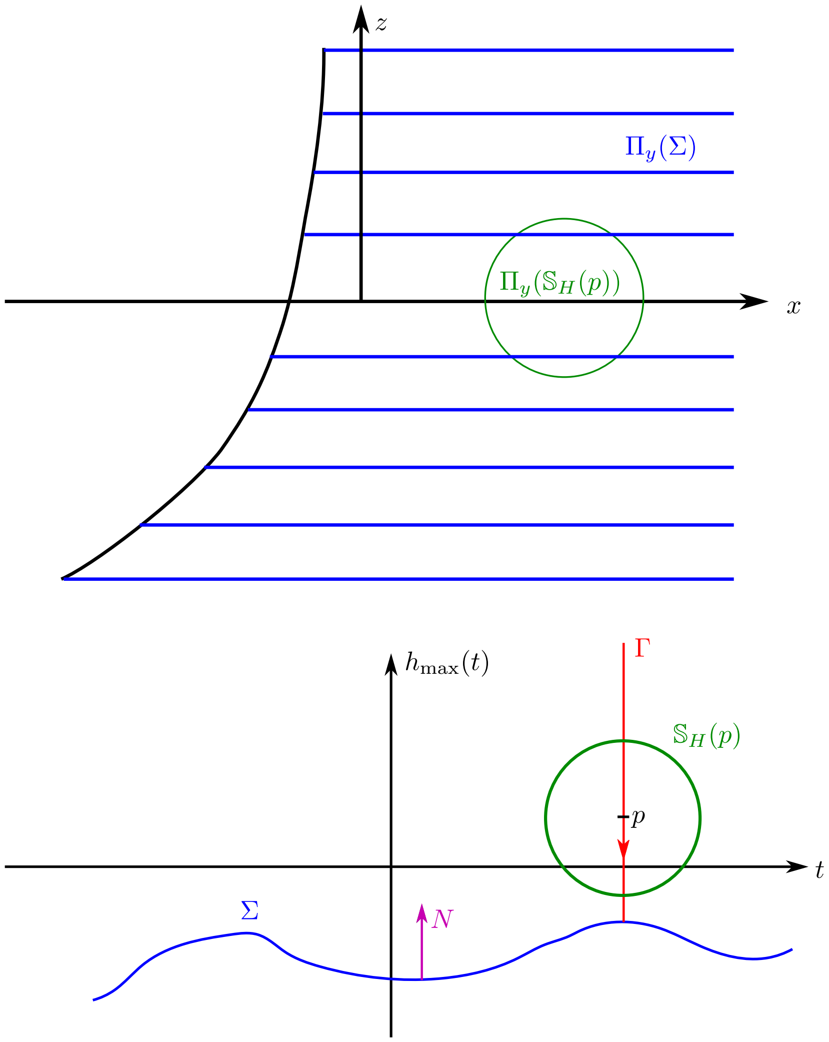


FIGURE 1.1. Lemma 1.3 (a): Comparison argument indicating that $\sup I_{\max} = \infty$ and $\inf I_{\max} = -\infty$ are impossible

(a): Assume $\sup I_{\max} = \infty$ or $\inf I_{\max} = -\infty$; say, without loss of generality, $\sup I_{\max} = \infty$. Consider a constant mean curvature H sphere $\mathbb{S}_H$ centred at $(0,0,0)$ in Sol_3 and let $\Pi_y\colon\ \mathrm{Sol}_3 \to \mathbb{R}^2$ be defined by $\Pi_y(x, y, z) := (x, z)$.

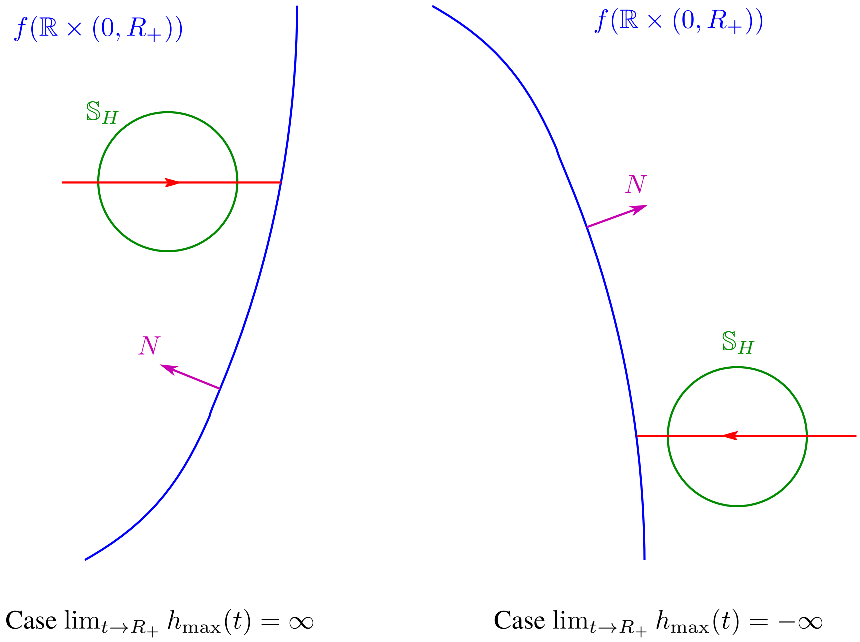


FIGURE 1.2. Geometry for Lemma 1.3 (b)

Due to our assumption and the compactness of $\mathbb{S}_H$ we can translate $\mathbb{S}_H$ to a sphere $\mathbb{S}_H(p)$ centred at a point p such that $\Pi_y(\mathbb{S}_H(p))$ is contained in $\Pi_y(\Sigma)$. Moving spheres along some y-axis Γ through p inside the mean convex side of Σ towards the surface leads to a first tangential contact point in the interior of Σ. The maximum principle then shows $\Sigma = \mathbb{S}_H$, which is a contradiction. See Figure 1.1 on page 9.

(b): If this were false, then h could only be unbounded for $t \to R_\pm$. Without loss of generality, $\lim_{t \to R_+} h(t) = \pm\infty$. Both cases are ruled out by moving spheres towards $f|_{\mathbb{R} \times (0, R_+)}$; compare with Figure 1.2 on page 10.

(c): Let $\tilde{F} \colon \mathbb{R}^3 \to \mathbb{R}^2$, $\tilde{F}(\tau, \xi, \eta) := (\eta, F(\tau, \xi, \eta))$, so that the maximal solution of $x'(t) = \tilde{F}(t, x(t))$ with $x(0) = (a, b)$ is given by $y(t) := (h(t), h'(t))$.

We know the phase space of $\tilde{F}$ is $\mathbb{R}^3$. General ODE theory implies that

$$I_{\max} \to \mathbb{R}^2, \qquad t \mapsto (t, y(t)) = (t, h(t), h'(t))$$

leaves every compact subset in $\mathbb{R}^3$, in particular $[R_-, R_+] \times [-K, K] \times [-C, C]$ for every $C > 0$. In view of (a) and (b) this implies $\lim_{t \to R_\pm} |h'(t)| = \infty$.

Let us now confirm the sign of $\lim_{t \to R_\pm} h'(t)$. On the contrary, suppose

$$\lim_{t \to R_+} h'(t) = -\infty$$

and consider

$$\alpha := \inf\{t \in (0, R_+) \colon h'|_{(t,R_+)} < 0\}.$$

For $t_0 \in (\alpha, R_+)$ let Γ be defined by

$$\Gamma \colon [t_0, +\infty) \to \mathrm{Sol}_3, \qquad s \mapsto (s, h(t_0), 0).$$

Moving spheres along Γ we get a first tangential point of contact in the interior of the surface $f|_{\mathbb{R} \times (\alpha, R_+)}$. The normals of $\mathbb{S}_H$ and $f|_{\mathbb{R} \times (\alpha, R_+)}$ coincide at this point because $\lim_{t \to R_+} h'(t) = -\infty$. Thus the maximum principle yields a contradiction; see Figure 1.3 on page 11. For $\lim_{t \to R_-} h'(t) = -\infty$ we argue similarly.

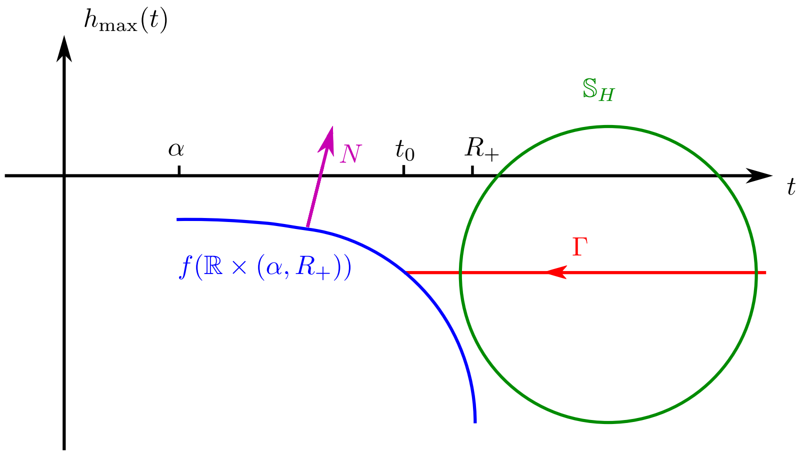


FIGURE 1.3. Situation for (c) from Lemma 1.3

(d): The existence of $t_0 \in I_{\max}$ with $h'(t_0) = 0$ is clear since h' is continuous and changes sign at least once by (c). Assume h were not strictly monotonically increasing on (t_0, R_+). Then $\beta := \sup\{t \in (t_0, R_+) \colon h'(t) \leq 0\}$ is strictly larger than t_0 and h defined on $[\beta, t_0]$. To rule out this case we apply the maximum principle to the surface

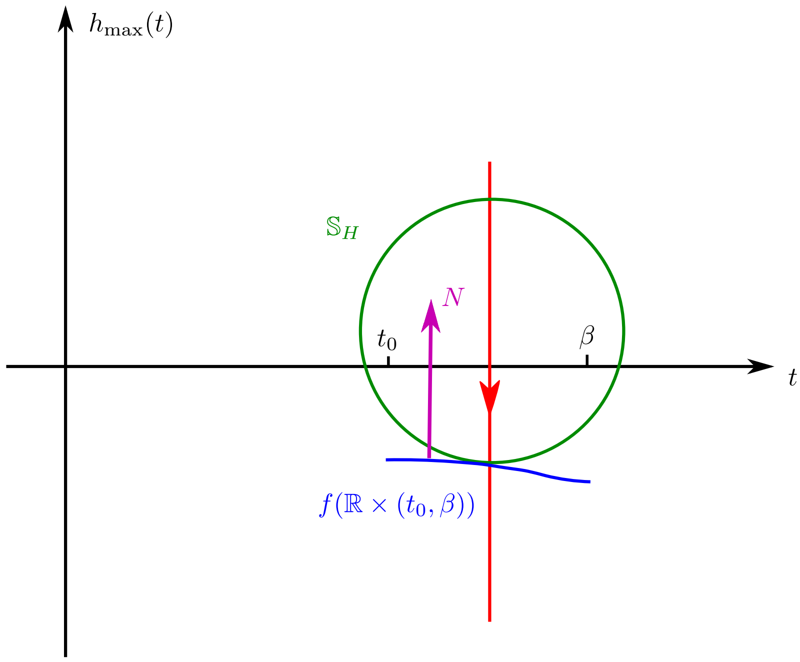


FIGURE 1.4. Situation for (d) from Lemma 1.3

$f|_{\mathbb{R}\times(t_0,\beta)}$, and move spheres to this surface having no boundary contact; see Figure 1.4 on page 12. We reason similarly for h on (R_-, t_0).

(e): It is easy to check $\sigma_{yz} \circ \Phi_s = \Phi_s \circ \sigma_{yz}$, so that a reflection of the solution through $\{x = 0\}$ gives another solution of the same ODE. If we assume $b = 0$, then the initial values $h(0) = a$ and $h'(0) = 0$ are invariant under σ_{yz}, so that we obtain the same solution. This proves $R_+ = -R_-$. $\square$

We are interested in a particular solution of the ODE, see Figure 1.5:

Proposition 1.4 (0-height solution)**.** *There is $a_0 \in \mathbb{R}$ such that $h(\pm R(a_0)) = 0$ for the maximal solution with $h(0) = a_0$ and $h'(0) = 0$. Furthermore we have $R(a_0) = -a_0$.*

Proof. The function $\varphi\colon \mathbb{R} \to \mathbb{R}$, $\varphi(a) := h(R(a))$ is continuous. For $a = 0$ monotonicity implies $\varphi(a) > 0$. If we had $\varphi(a) > 0$ for all $a \le 0$, then we could find $\tilde{a} < 0$ such that it were possible to move a sphere to the surface $f|_{\mathbb{R}\times(0,R(\tilde{a}))}$ without touching its

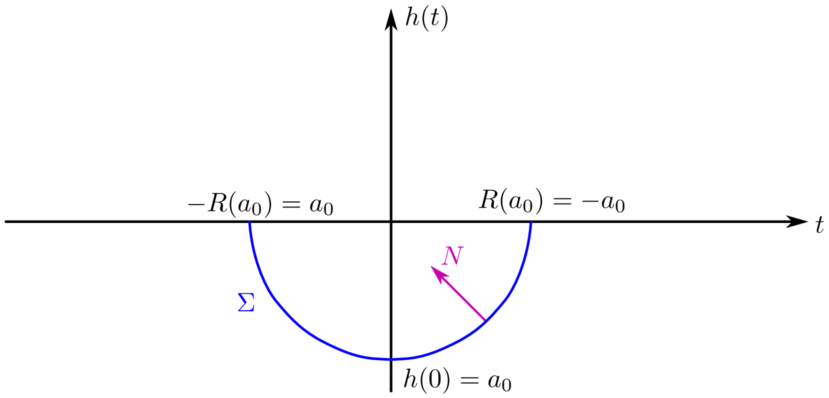


FIGURE 1.5. Illustration of Proposition 1.4

boundary, a contradiction. So there is some $\tilde{a} < 0$ with $\varphi(\tilde{a}) < 0$, and by the intermediate value theorem we get $a_0 \in (-\infty, 0)$ with $\varphi(a_0) = 0$.

To show $R(a_0) = -a_0$ let us consider $\tilde{f} := \rho_+ \circ f$. Then Proposition 1.2 (b) implies that h is also a 0-height solution to the initial values $h(0) = -R(a_0)$ and $h'(0) = 0$ with $I_{\max} = (a_0, -a_0)$. This can only hold for $R(a_0) = -a_0$. □

We use one 0-height solution to obtain a smooth embedded closed curve γ generating an invariant cylinder f with constant mean curvature $H > 0$ and get our first main result:

Theorem 1.5. *Consider the fibration* $\mathrm{Sol}_3 \to \mathbb{R}$*. Then for each* $H > 0$ *there is a smooth embedded simple closed curve* γ *in a* $\mathbb{R}^2$*-fibre which generates a* Γ*-invariant embedded surface* f *in* Sol_3 *with constant mean curvature* H*. The surface is invariant by a dihedral subgroup of order* 8*, generated by* $\{\sigma_{xz}, \sigma_{yz}, \rho_\pm\}$*.*

For the notation recall that Γ denotes the group of left-translations along the base in the model of Sol_3 described in Section 1.1; definitions of σ_{xz}, σ_{yz} and $\rho_\pm$ can be found there, too. In the following we refer to these surfaces as *MCH-cylinders with axis c in* Sol_3 or as *MCH-cylinders with basic axis in* Sol_3. We have computed an example, see Figure 1.6, and details of the computation are explained in Remark 1.7.

Proof. Let $h\colon (-R, R) \to \mathbb{R}$ be a 0-height solution. We have $\sigma_{xz} \circ \Phi_s = \Phi_s \circ \sigma_{xz}$, so that we can extend the surface by reflecting through $\{y = 0\}$. This extension gives rise to

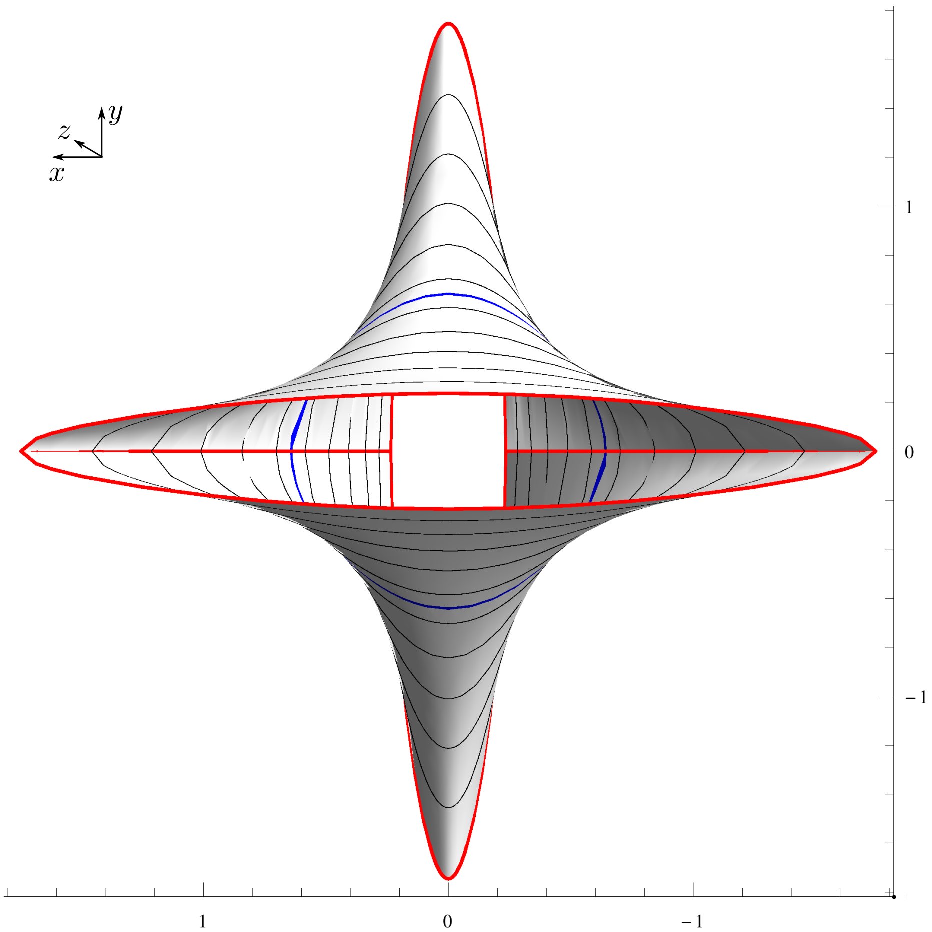


FIGURE 1.6. Computed example of an MC1-cylinder in Sol_3 established in Theorem 1.5. All level lines shown are the intersection of the cylinder with a $\mathbb{R}^2$-fibre of the fibration $\mathrm{Sol}_3 \to \mathbb{R}$, $(x, y, z) \mapsto z$. The level lines are isometric. The curve in blue is contained in the fibre $S_0 = \{z = 0\}$ and generates the MC1-cylinder.

a closed curve γ. The curve γ is smooth since h is asymptotic to a y-axis. Monotonicity of h implies embeddedness of γ. This proves the claim about the generating curve.

For the isometry group we note that invariance by σ_{xz} is obvious by construction of γ. The invariance by σ_{yz} follows from Lemma 1.3 (e). Similarly we argue for the invariance by $\rho_{\pm}$: Due to $R(a_0) = -a_0$, the initial values of the half-cylinder solutions remain invariant, hence we get the same solution. $\square$

Remark 1.6 (Uniqueness of 0-height solution)**.** We conjecture there is exactly one 0-height solution, but we do not have a proof at hand. If there were 0-height solutions h_0 and h_1 to initial values $h_0(0) = a_0$ and $h_1(0) = a_1$ respectively, then both would satisfy $R(a_0) = -a_0 \neq -a_1 = R(a_1)$. Then one solution would be above the other one, so one cylinder would be on the mean convex side of the other cylinder. It appears we could use the maximum principle to rule out this situation. However, we cannot exhibit a point of tangential contact by moving one solution along the y-axis because translations along the y-axis and translations along c do not commute. It seems we need a more elaborate application of the maximum principle – a *half-space theorem* – to rule out that an MCH-cylinder can be on the mean convex side of another MCH-cylinder. In order to apply the general half-space theorem by Mazet [**Maz13**] we have to verify two crucial assumptions: First, the parabolicity of our cylinders, that is, they must be conformal to a punctured plane. This assumption is satisfied in our case due to translational invariance. Second, there is an assumption on the mean curvature of equidistant surfaces to the given MCH-cylinder. It appears difficult to verify and we do not know whether it holds or not.

Remark 1.7 (Computed example)**.** We used Mathematica to calculate the MCH-cylinders with axis c. We have computed the ODE in Appendix A, see Proposition A.1 on page 93. We set $H = 1$. Upon iteration we calculated for $a := -0.642176$ that

$$h(R(a)) < 10^{-7} \qquad \text{and} \qquad R(a) = -a = 0.642176,$$

as expected by Proposition 1.4. Finally we extended the solution curve by a reflection through $\{y = 0\}$. See Figure 1.6.

[**Lop14**] also has a numerical example, but we believe it is less precise due to a different approach of exhibiting the initial value $h(0) = a$ numerically. For comparison, we note that $h(0) \approx -0.6425$ in [**Lop14**], which we consider a less precise value. For instance, it does not satisfy $R(a) = -a$ numerically and we get $h(R(a)) \approx 2 \cdot 10^{-4}$.

It is natural to look at the family of MCH-cylinders with $H \in (0, \infty)$. Computations with Mathematica, illustrated by Figure 1.7, are evidence for the following:

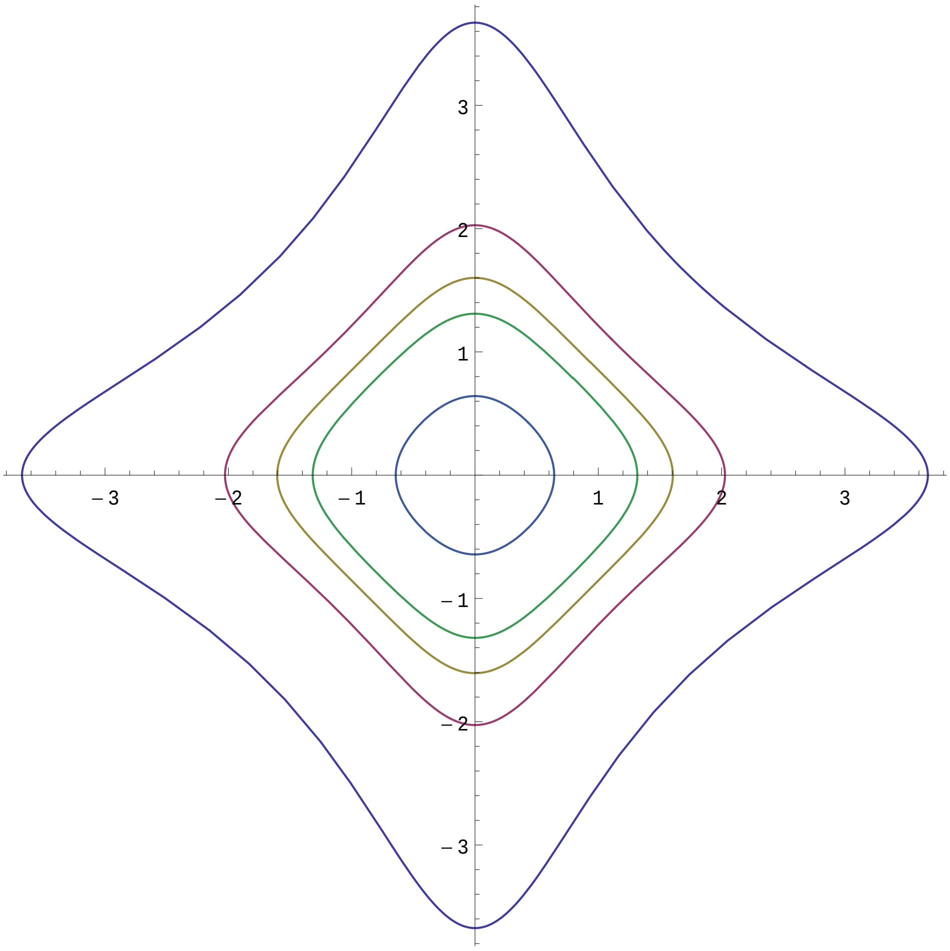


FIGURE 1.7. Generating curves of the MCH-cylinders of Theorem 1.5: From outer to inner contour the mean curvature H takes the values 0.5, 0.6, 0.65, 0.7 and 1.

Conjecture 1.8. *The* MCH*-cylinders with axis* c *form an analytic family in* $H \in (0, \infty)$. *For* $H \to 0$ *the surfaces are unbounded and for* $H \to \infty$ *they shrink to* c.

1.2.3. Conjecture on non-embedded solutions with axis c. A shooting method leads to computed examples of non-embedded MCH-cylinders with axis c in Sol_3. We shoot orthogonally from the diagonal c_+ and aim at the y-axis, compare with Figure 1.8. Assume the solution curve meets the y-axis at $T > 0$. Then $y'(T)$ determines the angle between γ and the y-axis. We extend this portion by a rotation of angle π about c_+ and reflections through $\{x = 0\}$ and $\{y = 0\}$ to a closed curve $\gamma = (x, y, 0)$. The resulting curve is built up from 8 such portions, possibly non-smooth at multiples of T.

Recall that the turning number $\mathrm{turn}(\gamma)$ satisfies

$$2\pi\,\mathrm{turn}(\gamma) = \int_0^{8T} \kappa_{\mathrm{eucl}}(\gamma)\,dt + \mathrm{ext}(\gamma),$$

where the second term $\mathrm{ext}(\gamma) = 8y'(T)$ denotes the sum of the exterior angles. If γ meets the y-axis orthogonally at T then γ is smooth and $\mathrm{ext}(\gamma) = 0$.

To compute examples we fix $H = 1$ and proceed as follows:

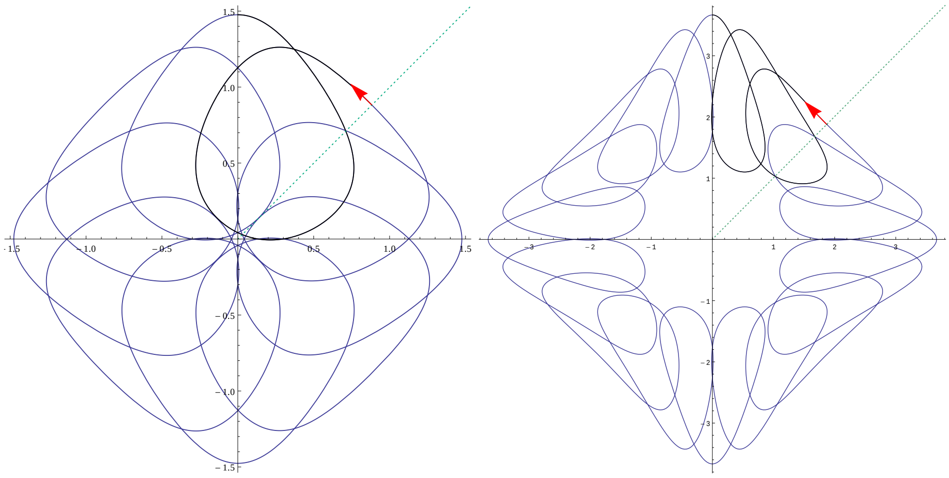
- Take $\gamma(0) = (d, d, 0)$ for some $d \in \mathbb{R}$ and $\gamma'(0) = \frac{1}{\sqrt{2}}(-1, 1, 0)$ as initial values.
- Suppose the resulting curve meets the y-axis at time $T = T(d) > 0$.
- Vary d while maintaining the same turning number of closed extension curve.
- Exhibit d_1 and d_2 with $y'(T(d_1)) < 0$ and $y'(T(d_2)) > 0$. An intermediate value argument gives some d_0 between d_1 and d_2 with $y'(T(d_0)) = 0$.

With this ansatz we computed solutions with turning number 9 and 17, shown in Figure 1.8a and Figure 1.8b.

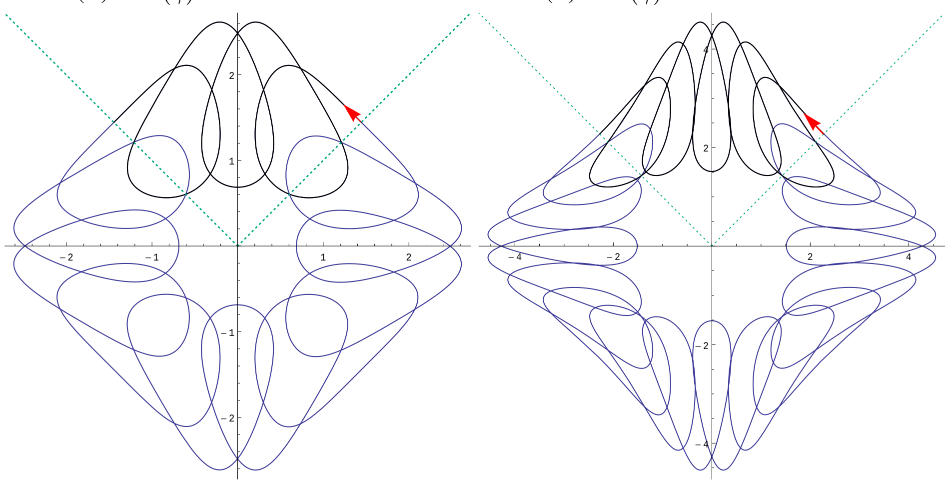
Aiming at the other diagonal c_- instead of the y-axis we find solutions with further turning numbers. See Figure 1.8c and Figure 1.8d for solutions with turning number 13 and 21.

It is straightforward to compute more examples with turning number $5 + 4k$ where $k \in \mathbb{N}$. The particular value $d = 0.429474$ corresponds to the solution generating the embedded cylinder.

Moreover, we computed an example with turning number 5 for the values $H = \frac{1}{2}$ and $d = -0.965$. Increasing H as well as d, we computed examples with turning number 5 up to $H = 0.759$. It appears that these solutions with turning number 5 degenerate to the fivefold cover of a cylinder solution for some $H_0 \in (0.759, 1)$; see Figure 1.9.

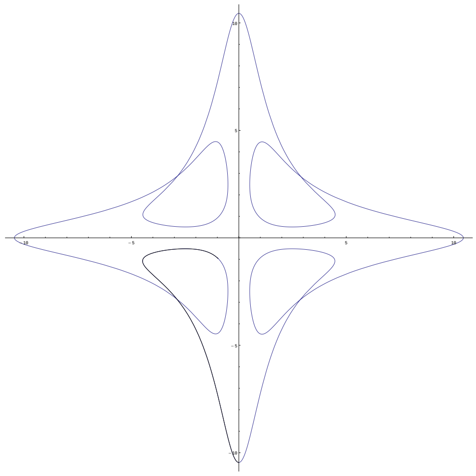


(A) $\mathrm{turn}(\gamma) = 9$ and $d = 0.8856$ (B) $\mathrm{turn}(\gamma) = 17$ and $d = 1.8755$

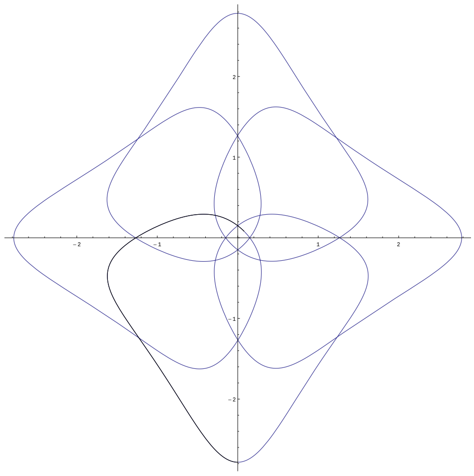


(C) $\mathrm{turn}(\gamma) = 13$ and $d = 1.445$ (D) $\mathrm{turn}(\gamma) = 21$ and $d = 2.277$

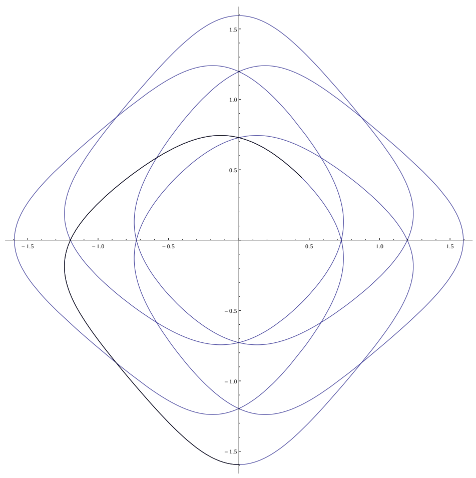
FIGURE 1.8. Computed examples of solution curves γ with turning number $5 + 4k$, where $k \in \{1, 2, 3, 4\}$. They generate non-embedded cylinders with $H = 1$. A fundamental portion of the curve is shown in black; it meets the dotted diagonals or the y-axis at right angle and generates the solution curve upon reflection.



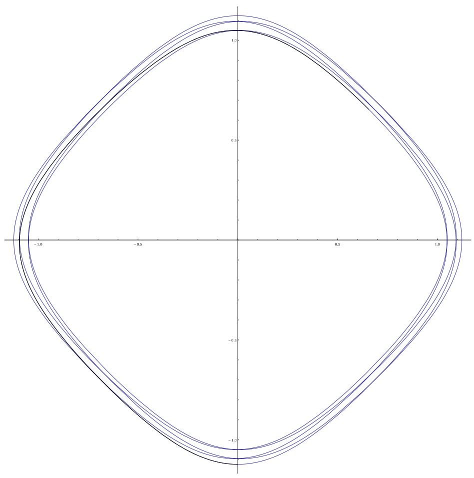
(A) $H = 0.5$ and $d = -0.965$



(B) $H = 0.7$ and $d = 0.08$



(C) $H = 0.75$ and $d = 0.445$



(D) $H = 0.759$ and $d = 0.655$

FIGURE 1.9. Solution curves with turning number 5 converge to a multiple cover of the embedded MCH-cylinder solution upon increasing H and d.

Conjecture 1.9. *For each $H > 0$ there is $m = m(H) \in \mathbb{N}$ such that for every natural number $k \geq 1$ there exists a non-embedded closed curve with turning number $m + 4k$ as generating curve of a Γ-invariant surface with constant mean curvature H.*

A proof of this conjecture seems beyond the techniques used in the present thesis.

1.3. Surfaces invariant under translations along $c_\pm$

For constant mean curvature surfaces invariant under $\Gamma_\pm$ we will proceed as for those invariant under Γ.

1.3.1. ODE for surfaces invariant by translations along $c_\pm$. For our second surface family we can consider the foliation $(S_{\pm,s})_{s\in\mathbb{R}}$ of planes above diagonals in the x-y-plane. We have

$$S_{\pm,s} = \left\{ \Big(\frac{x}{\sqrt{2}}, \mp \frac{x}{\sqrt{2}}, z \Big) : x, z \in \mathbb{R} \right\}, \quad s \in \mathbb{R}.$$

Obviously, this foliation is invariant by translations along $c_\pm$. For surfaces invariant by $\Gamma_\pm$, a discussion as in Proposition 1.2 gives the following result for the ODE of graphical solutions:

Proposition 1.10. *Let $H \in \mathbb{R}$. There is a smooth function $F\colon \mathbb{R}^3 \to \mathbb{R}$ such that the invariant surface*

$$f\colon \mathbb{R} \times J \to \mathrm{Sol}_3, \qquad f(s,t) := \Phi_{\pm,s}\Big(\frac{t}{\sqrt{2}}, \mp \frac{t}{\sqrt{2}}, h(t) \Big) \qquad \text{where } h \in \mathcal{C}^2(J, \mathbb{R}),$$

has constant mean curvature H with respect to the upper normal if and only if

$$h''(t) = F(t, h(t), h'(t)) \qquad \text{for all } t \in J. \tag{1.5}$$

1.3.2. MCH-cylinders with axis $c_\pm$. We recall that we consider the metric Lie group Sol_3 as $\mathbb{R}^3$ with left-invariant Riemannian metric $\langle \cdot, \cdot \rangle = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2$ and let $\Gamma_\pm$ be the family of left-translations along $c_\pm \colon \mathbb{R} \to \mathrm{Sol}_3$, $c(s) = \Big(\frac{s}{\sqrt{2}}, \pm \frac{s}{\sqrt{2}}, 0 \Big)$. We obtain the following result:

Theorem 1.11. *For each $H > 0$ there is a smooth embedded simple closed curve γ in $S_{\pm,0} = \Big\{ \big(\frac{x}{\sqrt{2}}, \mp \frac{x}{\sqrt{2}}, z \big) : x, z \in \mathbb{R} \Big\}$ which generates a $\Gamma_\pm$-invariant embedded surface f in Sol_3 with constant mean curvature H. It is invariant by ρ_+ and ρ_-.*

We call this surface *MCH-cylinder with axis $c_\pm$*.

Proof. The discussion from Subsection 1.2.2 is also applicable to $\Gamma_\pm$-invariant surfaces, so that we only indicate the differences in the proof.

Consider a maximal solution of (1.5). In order to obtain a symmetric solution as in Lemma 1.3 (e) we fix the initial value at $h'(0) = 0$ and argue as follows: Rotations of angle π about c_+ and c_- commute with translation along $c_\pm$. This shows the symmetry in this case, the other items are proved in the same way.

For the 0-height solution in this case we argue exactly as in Proposition 1.4. Since $\rho_\pm$ commutes with translation along $c_\pm$ we can extend a 0-height solution to a smooth embedded closed curve. This finishes the proof. $\square$

Remark 1.12. Constant mean curvature surfaces invariant under $\Gamma_\pm$ in Sol_3 have not been considered before and this result shows that $\Gamma_\pm$ generates interesting surfaces. We have not computed the ODE for these surfaces so that we do not make any claims about non-embedded solutions with axis $c_\pm$.

We believe Conjecture 1.8 also applies to embedded MCH-cylinders with axis $c_\pm$:

Conjecture 1.13. *The MCH-cylinders with axis $c_\pm$ form an analytic family with respect to $H \in (0, \infty)$. For $H \to 0$ the surfaces are unbounded and for $H \to \infty$ they shrink to c.*

CHAPTER 2

$_{\rm MC}H$-cylinders in non-compact $E(\kappa, \tau)$-spaces

The $E(\kappa, \tau)$-spaces are Riemannian fibrations $E \to B$ with geodesic fibres, bundle curvature $\tau \in \mathbb{R}$ and base curvature $\kappa \in \mathbb{R}$. In Section 2.1 we describe these spaces. Most results concerning constant mean curvature surfaces then become “horizontal“ or “vertical“ generalisations of results known for the case of $\mathbb{R}^3$. It turns out that the arguments given in Sections 1.2 and 1.3 carry over to prove existence of *tilted* $_{\rm MC}H$*-cylinders* in $E(\kappa, 0)$ and of *horizontal* $_{\rm MC}H$*-cylinders* in $E(\kappa, \tau)$ for $\tau \neq 0$. In the final section we compute the horizontal diameter of a horizontal $_{\rm MC}H$-cylinder in $E(\kappa, \tau)$-spaces with base curvature $\kappa \leq 0$.

2.1. Non-compact $E(\kappa, \tau)$-spaces

The $E(\kappa, \tau)$-spaces are simply connected homogeneous 3-manifolds diffeomorphic to $\mathbb{R}^3$, $\mathbb{S}^3$ or $\mathbb{S}^2 \times \mathbb{R}$ and arise as Riemannian fibrations $E \to B$ with geodesic fibres, where B has curvature $\kappa \in \mathbb{R}$ and the bundle curvature is $\tau \in \mathbb{R}$. Because we exclude the Berger spheres, that is $\kappa > 0$ and $\tau \neq 0$ arbitrary, we may assume $E = B \times \mathbb{R}$.

2.1.1. General properties. The $E(\kappa, \tau)$-spaces have some geometric properties, which can be stated without an explicit model.

Geodesics. In $E = B \times \mathbb{R}$ the *vertical translations*

$$T_s \colon B \times \mathbb{R} \to B \times \mathbb{R}, \qquad T_s(p, t) = (p, t + s), \qquad s \in \mathbb{R},$$

are isometries, giving rise to a Killing field ξ. As a consequence of *Clairaut's Theorem*, geodesics have the following property which is also true for compact $E(\kappa, \tau)$-spaces:

Proposition 2.1 ([**Eng06**, Lemma 3.7])**.** *Let* $c \colon \mathbb{R} \to E(\kappa, \tau)$ *be a unit-speed geodesic. Then there is* $\alpha \in [0, \pi]$ *with* $\langle c', \xi \circ c \rangle \equiv \cos(\alpha)$. *We call* α slope of c with respect to ξ. *The projection* $\tilde{c} := \Pi \circ c$ *is a curve of constant geodesic curvature* $-\tau \cot(\alpha)$ *in* B.

We call the geodesic fibres, corresponding to $\alpha = 0$, *vertical geodesics*. They admit arbitrary rotations as isometries. On the other hand, the case of $\alpha = \frac{\pi}{2}$ corresponds to *horizontal geodesics*. They admit rotations by an angle π.

For $\tau = 0$ geodesics project to geodesics of the base space B, while for $\tau \neq 0$ only horizontal geodesics project to geodesics of B. In the following we are only considering geodesics c in E which project onto geodesics in B and have slope $\alpha \in (0, \frac{\pi}{2}]$.

Foliation by vertical planes and induced translations. Let c be a geodesic whose projection $\tilde{c} := \Pi \circ c$ is also geodesic. Then there is a foliation of B or of $B = \mathbb{S}^2$ minus two points by geodesics $(\tilde{\gamma}_s)_{s\in\mathbb{R}}$ perpendicular to $\tilde{c}$ such that $\tilde{\gamma}_s(0) = \tilde{c}(s)$ for all $s \in \mathbb{R}$. The vertical planes $P_s := (\tilde{\gamma}_s \times \mathbb{R})_{s\in\mathbb{R}}$ are therefore a foliation of E respectively $\mathbb{S}^2 \times \mathbb{R}$ minus two vertical lines. As $\alpha \neq 0$ the geodesic c meets each P_s transversally.

Geodesics in $E(\kappa,\tau)$-spaces are orbits of one-parameter families of isometries, for a proof see [**Eng06**, Theorem 2.5]. Such a one-parameter family can be chosen as follows: In the base B, let $\big(\tilde{\psi}_s\big)_{s\in\mathbb{R}}$ be the family of translations along $\tilde{c}$ with $\tilde{\psi}_s(\tilde{c}(0)) = \tilde{c}(s)$. By [**Man14**, Corollary 2.11] we can lift each $\tilde{\psi}_s$ horizontally and obtain an orientation-preserving isometry $\psi_s \colon E \to E$. Vertical translations T_σ commute with ψ_s so that we can consider

$$\Phi_s := \psi_{s\sin(\alpha)} \circ T_{s\cos(\alpha)}$$

This defines a one-parameter family of isometries $\Gamma := (\Phi_s)_{s\in\mathbb{R}}$ in E, which by construction satisfies $\Phi_s(c(0)) = c(s)$. We refer to the isometries as *translations along* c, and list some straightforward properties:

Proposition 2.2. *Let c be a geodesic in $E(\kappa,\tau)$ with geodesic projection $\tilde{c}$ and let Γ be the family of translations along c.*

(a) For $\tau = 0$ reflection through $\tilde{c} \times \mathbb{R}$ is an isometry commuting with Γ.

(b) For $\tau \neq 0$ the geodesic c is horizontal and rotation of angle π about c commutes with the family Γ.

(c) For any $\tau \in \mathbb{R}$ we have the following:
- *The horizontal lift γ of $\tilde{\gamma}_0$ with $\gamma(0) = c(0)$ is a horizontal geodesic and the rotation of angle π about γ, denoted by ρ, satisfies $\rho \circ \Phi_s = \Phi_{-s} \circ \rho$.*
- *Vertical translations and Γ commute.*

Constant mean curvature spheres. We quote various results on MCH-spheres in non-compact $E(\kappa,\tau)$-spaces. First we introduce the so-called *magic number*:

Definition (Magic number)**.** Let E be an $E(\kappa,\tau)$-space. We call

$$H(E) := \inf\{H > 0\colon \text{there exists an immersed MC}H\text{-sphere in } E\}$$

magic number of E.

In the non-compact case we have

$$H(E(\kappa,\tau)) = \begin{cases} \frac{\sqrt{-\kappa}}{2} & \text{if } \kappa \le 0, \\ 0 & \text{if } \kappa > 0. \end{cases}$$

MCH-spheres for $H > H(E)$ in a non-compact $E(\kappa,\tau)$-space are

- invariant by rotation about a fibre,
- unique up to isometries and
- embedded.

By [**AR05**, Theorem 6], any immersed constant mean curvature sphere in a non-compact $E(\kappa,\tau)$-space is a rotational sphere, which also implies uniqueness up to isometries. Explicit examples of rotationally invariant MCH-spheres in $\mathbb{H}^2 \times \mathbb{R}$, $\mathbb{S}^2 \times \mathbb{R}$, Nil_3 and $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$ can be found in the following papers: [**Onn08, ST05, HH89, FMP99, Pen10**].

The assumption on E being non-compact is crucial for embeddedness. Examples of non-embedded rotationally invariant MCH-spheres in the Berger spheres have been found by Torralbo in [**Tor10**].

2.1.2. $E(\kappa,\tau)$**-spaces with** $\kappa \le 0$**.** The $E(\kappa,\tau)$-spaces with $\kappa \le 0$ are simply connected homogeneous 3-manifolds diffeomorphic to $\mathbb{R}^3$ and arise as metric Lie groups. We describe a model and some geometric properties. The advantage of this model is that the limits $\kappa \to 0$ and $\tau \to 0$ are well-defined, also on the level of orthonormal frames.

Model. For our purpose the classification of [**MP12**] provides a convenient description of these spaces. For $\kappa \le 0$ and $\tau \in \mathbb{R}$ let

$$A(\kappa,\tau) := \begin{pmatrix} \sqrt{-\kappa} & 0 \\ 2\tau & 0 \end{pmatrix}.$$

We want to compute

$$\begin{pmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{pmatrix} := e^{zA(\kappa,\tau)}.$$

For $\kappa < 0$ we have

$$e^{zA(\kappa,\tau)} = \begin{pmatrix} e^{z\sqrt{-\kappa}} & 0 \\ \frac{2\tau}{\sqrt{-\kappa}}\left(e^{z\sqrt{-\kappa}}-1\right) & 1 \end{pmatrix}$$

and for $\kappa = 0$ we get

$$e^{zA(0,\tau)} = \begin{pmatrix} 1 & 0 \\ 2\tau z & 0 \end{pmatrix}.$$

We observe $\lim_{\kappa\to 0} e^{zA(\kappa,\tau)} = e^{zA(0,\tau)}$ for all $z,\tau\in\mathbb{R}$ so that the first expression also makes sense for $\kappa = 0$.

The space $\mathbb{R}^2 \ltimes_{A(\kappa,\tau)} \mathbb{R}$ is a metric Lie group with group structure

$$(x_1,y_1,z_1) * (x_2,y_2,z_2) := \Big((x_1,y_1) + e^{z_1A(\kappa,\tau)}(x_2,y_2), z_1+z_2\Big)$$

and Riemannian metric

$$\begin{aligned}\langle\cdot,\cdot\rangle_{(x,y,z)} = &\left(e^{-2z\sqrt{-\kappa}} - \frac{4\tau}{\kappa}(e^{-z\sqrt{-\kappa}}-1)^2\right) dx^2 + dy^2 + dz^2 \\ &+ \frac{2\tau}{\sqrt{-\kappa}}(e^{-z\sqrt{-\kappa}}-1)(dx\otimes dy + dy\otimes dx).\end{aligned} \tag{2.1}$$

The canonical orthonormal frame, obtained by left-translation of the Euclidean frame from the origin $(0,0,0)$, is

$$E_1(x,y,z) = e^{z\sqrt{-\kappa}}\partial_x + \frac{2\tau}{\sqrt{-\kappa}}\left(e^{z\sqrt{-\kappa}}-1\right)\partial_y,\ \ E_2(x,y,z) = \partial_y,\ \ E_3(x,y,z) = \partial_z.$$

The Riemannian connection with respect to this frame has the following representation:

$$\begin{array}{lll} \nabla_{E_1}E_1 = \sqrt{-\kappa}E_3, & \nabla_{E_1}E_2 = \tau E_3, & \nabla_{E_1}E_3 = -\sqrt{-\kappa}E_1 - \tau E_2, \\ \nabla_{E_2}E_1 = \tau E_3, & \nabla_{E_2}E_2 = 0, & \nabla_{E_2}E_3 = -\tau E_1, \\ \nabla_{E_3}E_1 = \tau E_2, & \nabla_{E_3}E_2 = -\tau E_1, & \nabla_{E_3}E_3 = 0. \end{array}$$

Proposition 2.3. *Let $\kappa \le 0$ and $\tau\in\mathbb{R}$. On $\mathbb{R}^2$ we consider the Riemannian metric*

$$\tilde g_{(x,z)} = e^{-2z\sqrt{-\kappa}}dx^2 + dz^2.$$

Then $\Pi\colon (\mathbb{R}^3,\langle\cdot,\cdot\rangle) \to (\mathbb{R}^2,\tilde g)$, $(x,y,z)\mapsto (x,z)$ is a Riemannian submersion with geodesic fibres over the simply connected surface $(\mathbb{R}^2,\tilde g)$ with constant curvature κ. This submersion has bundle curvature τ, so that $(\mathbb{R}^3,\langle\cdot,\cdot\rangle)$ is isometric to $E(\kappa,\tau)$.

Sketch of proof. We can refer to various Theorems in [**MP12**], but let us give the explicit argument:

- The vertical space is spanned by E_2 while the horizontal space is spanned by E_1 and E_3.
- For a horizontal vector $v = \lambda E_1 + \mu E_3$ we have $\tilde g_{(x,z)}(d\Pi\, v, d\Pi\, v) = \lambda^2 + \mu^2$, so that Π is indeed a Riemannian submersion.
- In view of the Riemannian connection we have $\langle R(E_1,E_3)E_3,E_1\rangle = \kappa - 3\tau^2$, so that $(\mathbb{R}^2,\tilde g)$ is a simply connected surface with constant curvature κ.
- We also have $\frac{1}{2}\langle \nabla_{E_3}E_1 - \nabla_{E_1}E_3, E_2\rangle = \tau$, which proves the claim about the bundle curvature. $\square$

A horizontal geodesic and the induced translations. The unit-speed curve

$$c\colon E\to E, \qquad c(s) := (0,0,s)$$

is a horizontal geodesic.

Our model is a metric Lie group and so a one-parameter family of isometries is

$$\Phi_s\colon E\to E, \qquad \Phi_s(x,y,z) := \left(e^{s\sqrt{-k}}x, \frac{2\tau}{\sqrt{-\kappa}}(e^{s\sqrt{-k}}-1)x+y, z+s\right),$$

which preserves c. We refer to $\Gamma := (\Phi_s)_{s\in\mathbb{R}}$ as *translations along* c. The infinitesimal generator or Killing field of Γ at $(x,y,z)\in E$ is given by

$$K_{(x,y,z)} = \frac{d}{ds}\Phi_s(x,y,z) = x\sqrt{-\kappa}\, e^{-z\sqrt{-\kappa}}xE_1 + 2\tau x e^{-z\sqrt{-\kappa}}E_2 + E_3. \tag{2.2}$$

We observe K is independent of y.

Foliation by vertical planes. We want to exhibit vertical planes $(P_s)_{s\in\mathbb{R}}$ as in Subsection 2.1.1. In fact, for the vertical planes $(P_s)_{s\in\mathbb{R}}$ with $P_s = \Pi^{-1}(\tilde\gamma_s)$ we will only need the curve $\tilde\gamma_0$ explicitly.

Proposition 2.4. *Consider*

$$\tilde\gamma_0\colon \mathbb{R}\to\mathbb{R}^2, \qquad \tilde\gamma_0(t) := \begin{cases} \left(\frac{\tanh(t\sqrt{-\kappa})}{\sqrt{-\kappa}}, \frac{\log(\operatorname{sech}(t\sqrt{-\kappa}))}{\sqrt{-\kappa}}\right) & \textit{for } \kappa<0,\\ (t,0) & \textit{for } \kappa = 0.\end{cases} \tag{2.3}$$

Then $\tilde\gamma_0(t)$ is also a continuous function of κ: For each $t\in\mathbb{R}$ the limit of $\tilde\gamma_0(t)$ for $\kappa<0$ and $\kappa\to 0$ exists and equals $(t,0)$. Moreover $\tilde\gamma_0$ is a unit-speed geodesic in $\mathbb{R}^2$ with respect to the metric induced by the Riemannian submersion $\Pi\colon \mathbb{R}^2\ltimes_{A(\kappa,\tau)}\mathbb{R}\to\mathbb{R}^2$. Each horizontal lift γ of $\tilde\gamma_0$ satisfies

$$\gamma'(t) = \operatorname{sech}\left(t\sqrt{-\kappa}\right)E_1 - \tanh\left(t\sqrt{-\kappa}\right)E_3. \tag{2.4}$$

Sketch of proof. The claim about the continuity of $\tilde{\gamma}_0(t)$ is clear.

For $\kappa = 0$ we have $\gamma_0(t) = (t, 0)$ and the metric induced on $\mathbb{R}^2$ is the Euclidean one, so that $\tilde{\gamma}_0$ is geodesic.

For $\kappa < 0$ we consider the upper half-plane $\mathbb{H} := \{(u, v)\colon v > 0\}$ and note that

$$g_{(u,v)} := \frac{1}{-\kappa v^2}\langle\cdot,\cdot\rangle_{\mathbb{R}^2}$$

defines a metric of constant sectional curvature κ on $\mathbb{H}$. Then

$$\mathbb{R} \to \mathbb{H}, \qquad t \mapsto \big(\tanh\big(t\sqrt{-\kappa}\big), \operatorname{sech}\big(t\sqrt{-\kappa}\big)\big)$$

parametrises a unit-speed geodesic semi-circle through $(0, 1)$. One can check that

$$\varphi\colon \mathbb{R}^2 \to \mathbb{H}, \qquad \varphi(x, z) := \Big(x\sqrt{-\kappa}, e^{z\sqrt{-\kappa}}\Big)$$

is an isometry with

$$\varphi^{-1}\colon \mathbb{H} \to \mathbb{R}^2, \qquad \varphi^{-1}(u, v) = \left(\frac{u}{\sqrt{-\kappa}}, \frac{\log(v)}{\sqrt{-\kappa}}\right).$$

Applying φ^{-1} to the geodesic in $\mathbb{H}$ proves the claim about $\tilde{\gamma}_0$. Regarding the horizontal lift γ we observe the following for $v := \operatorname{sech}\big(t\sqrt{-\kappa}\big)E_1 - \tanh\big(t\sqrt{-\kappa}\big)E_3$:

- v is horizontal,
- $\nabla_v v \equiv 0$ and
- $d\Pi\, v \equiv \tilde{\gamma}_0'$.

This completes the proof. □

2.2. Translationally invariant cylinders as ODE solutions

In this section we carry over the arguments used in Section 1.2 of the first chapter:

- As in case of Sol_3 we consider translationally invariant surfaces whose generating curves are graphical.
- The geometric discussion of the ODE for the graphical solution and its extension to a simple closed embedded curve carry over from Sol_3 almost literally, so that we only state what is different.

2.2.1. ODE for translationally invariant surfaces of constant mean curvature. The foliation by vertical planes $(P_s)_{s\in\mathbb{R}}$ is preserved by Γ. For $\mathcal{C}^2$-functions $x, y\colon J \to \mathbb{R}$ consider the unit-speed curve $\beta\colon J \to E$, $\beta(t) := T_{y(t)}\big(\gamma(x(t))\big)$, which is contained in the vertical plane P_0. A surface invariant by translation along c is parametrised by

$$f\colon \mathbb{R}\times J \to E, \qquad f(s,t) := \Phi_s(\beta(t)). \tag{2.5}$$

We specialise to $x(t) = t$ and $h(t) = y(t)$, that is, we are considering vertical graphs over γ. For these vertical graphs over γ we study the ODE for constant mean curvature:

Proposition 2.5. *Let H be in $\mathbb{R}$. There exists a smooth function $F\colon \mathbb{R}^2 \to \mathbb{R}$ such that the invariant surface*

$$f\colon \mathbb{R}\times J \to E, \qquad f(s,t) := \Phi_s\big(T_{h(t)}(\gamma(t))\big), \qquad \textit{where } h \in \mathcal{C}^2(J,\mathbb{R}),$$

has constant mean curvature H with respect to the upper normal if and only if

$$h''(t) = F(t, h'(t)) \qquad \textit{for all } t \in J. \tag{2.6}$$

Proof. Let $v_1 := \partial_s f$ and $v_2 := \partial_t f$. We denote the upper normal to f by N, so that $g_{ij} := \langle v_i, v_j\rangle$ and $b_{ij} := \langle \nabla_{v_i} v_j, N\rangle$ for $i,j \in \{1,2\}$ are the coefficients of the first and second fundamental form. Then the mean curvature of f is given by

$$H = \frac{b_{11}g^{11} + 2b_{12}g^{12} + b_{22}g^{22}}{2}.$$

Here we note that H depends on t, $h'(t)$ and $h''(t)$, but not on $h(t)$ itself. This is due to the existence of vertical translations commuting with Γ.

We assume H to be constant and therefore get an implicit differential equation depending on $h'(t)$ and $h''(t)$. Now we want to show we can solve this implicit equation for $h''(t)$. We have

$$v_2 = \gamma' + h'\xi \quad \text{and} \quad \nabla_{v_2} v_2 = \underbrace{\nabla_\xi \gamma' + h'\nabla_{\gamma'}\xi}_{=:w} + h''\xi.$$

We obviously have $w = w(t, h'(t))$ and so the only term containing $h''(t)$ is

$$\frac{b_{22}g_{11}}{2\det(g)} = \frac{\langle \nabla_{v_2} v_2, N\rangle g_{11}}{2\det(g)} = \frac{\left(h''\langle N,\xi\rangle + \langle w, N\rangle\right) g_{11}}{2\det(g)}.$$

The surface f is a Killing graph with respect to the Killing field ξ, so that $\langle N, \xi\rangle$ is positive for N is the upper normal. We also have $g_{11} > 0$ since Γ does never act trivially.

Hence we can solve the implicit equation for h'' and obtain a function $F\colon \mathbb{R}^2 \to \mathbb{R}$ with $h''(t) = F(t, h'(t))$. This function F is smooth because each Φ_s is smooth and thus are g and b. It is defined on whole $\mathbb{R}^2$ because we can prescribe any kind of function $h\colon J \to \mathbb{R}$. □

2.2.2. Geometric discussion of the ODE: Half-cylinder solution and its extension to an embedded cylinder. The following lemma corresponds to Lemma 1.3:

Lemma 2.6 (Half-cylinder solution)**.** *Given $a, b \in \mathbb{R}$ and $H > H(E)$, the Picard-Lindelöf Theorem gives a unique maximal solution $h\colon I_{\max} \to \mathbb{R}$ with $h(0) = a$ and $h'(0) = b$ satisfying* (2.6)*. For each $a, b \in \mathbb{R}$ it has the following properties:*

(a) [Horizontal boundedness]: *There are real numbers $R_\pm = R_\pm(a,b)$ with $R_- < R_+$ such that $I_{\max} = (R_-, R_+)$.*
(b) [Vertical boundedness]: *There is $K = K(a,b) > 0$ such that $\lim_{t\to R_\pm} |h(t)| \leq K$.*
(c) [Asymptotic behaviour]: *We have $\lim_{t\to R_\pm} h'(t) = \pm\infty$.*
(d) [Monotonicity]: *There is $t_0 \in (R_-, R_+)$ with $h'(t_0) = 0$. On (R_-, t_0) the function h is monotonically decreasing and on (t_0, R_+) it is monotonically increasing.*
(e) [Symmetry]: *For $b = 0$ we have $R := R_+ = -R_-$ for the maximal solution with $h(0) = a$ and $h'(0) = 0$.*

Proof. Items (a) to (d) can be proved as in the case of Sol_3, see Lemma 1.3.

(e): For $b = 0$ the tangent $\beta'(0)$ is horizontal. If $\tau = 0$ then reflection through the vertical plane $\tilde{c} \times \mathbb{R}$ is an isometry. This reflection and Γ commute and so the reflected graph satisfies the same ODE. Moreover the initial values are invariant. Hence the reflection leaves the solution invariant.

In case $\tau \neq 0$ let us translate the graph such that $a = 0$. Applying a rotations of angle π about c and γ yields a graphical solution satisfying the same ODE (by Proposition 2.2 (b) and (c)) and the initial values remain invariant under these rotations. In both cases we conclude $R_+ = -R_-$. □

In case of Sol_3 we needed a 0-height solution because the ODE (1.2) depends also on $h(t)$. In the present situation the symmetric solution from (e) generates a *horizontal cylinder* for all τ and a cylinder with sloped axis for $\tau = 0$, that is, the following main result includes *tilted* MC*H-cylinders* in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$:

Theorem 2.7. *Let c be a geodesic in a non-compact $E(\kappa,\tau)$-space with geodesic projection, that is, c has slope $\alpha \in (0, \frac{\pi}{2}]$ for $\tau = 0$ and $\alpha = \frac{\pi}{2}$ for $\tau \neq 0$; let Γ be the family of translations along c. For each $H > H(E)$ there is a smooth embedded simple closed curve β which generates a Γ-invariant cylinder f with constant mean curvature H. The surface f is embedded, except in $\mathbb{S}^2 \times \mathbb{R}$.*

For arbitrary τ the surface is invariant by a rotation of angle π about γ. For $\tau = 0$ the surface has a vertical mirror plane containing the axis c. If the axis is horizontal the surface is invariant by a rotation of angle π about its axis.

Proof. Let $h\colon (-R,R) \to \mathbb{R}$ be the symmetric solution from Lemma 2.6, i.e., $h'(0) = 0$. After a vertical translation we may assume $h(R) = h(-R) = 0$: in view of Lemma 2.6 (c) the graph meets γ orthogonally at $t = \pm R$.

We extend the graph h by ρ to a closed curve β, where ρ denotes the rotation of angle π about γ. The curve β is smooth because of the graph's asymptotic behaviour and monotonicity of h implies embeddedness of β. Due to $\rho \circ \Phi_s = \Phi_{-s} \circ \rho$ from Proposition 2.2 the curve β is generating a translationally invariant surface with constant mean curvature. The surface is embedded except for $\mathbb{S}^2 \times \mathbb{R}$ where translations can also be screw-motions. The claimed symmetries follow from Proposition 2.2 (a) and (b). $\square$

Remark 2.8. In his PhD thesis [**Pen10**], Penafiel studied various invariant surfaces in $E(-1,\tau) = \widetilde{\mathrm{PSL}}_2(\mathbb{R})$. A one-parameter family he considers is translation along a horizontal geodesic in $E(-1,\tau)$. He chose the upper-half plane model and the vertical plane containing the generating curve γ is

$$P_0 = \{(\cos(\theta), \sin(\theta), h)\colon \theta \in (0,\pi) \text{ and } h \in \mathbb{R}\}.$$

He considers graphs $h = h(\theta)$ generating an invariant MCH-surface. A flux computation, see [**Pen10**, Lemma 8.1.2], yields the representation

$$h(\theta) = \int \frac{(d - 2H\cot(\theta))\sqrt{1 + 4\tau^2\cos^2(\theta)}}{\sqrt{1 - \sin^2(\theta)(d - 2H\cot(\theta))^2}}\, d\theta - 2\tau\theta,$$

where d is a real number.

For some values of H, τ and d the integral can be computed explicitly, but for $H > \frac{1}{2}$ and $\tau \neq 0$ it seems that it has not been the case. With the help of Mathematica it is possible to represent h in terms of elliptic integrals, though.

2.3. Horizontal diameter of an MCH-cylinder with horizontal axis

We conclude this chapter with an application of the weight formula, i.e. by a flux computation: Without calculating the ODE for translationally invariant surfaces it is possible to determine $I_{\max}$ and thus the horizontal diameter of a horizontal MCH-cylinder. We exclude $\mathbb{S}^2 \times \mathbb{R}$ from the present discussion, that is, we are only considering $E(\kappa,\tau)$-spaces with $\kappa \leq 0$.

The graphs h, used to prove Theorem 2.7, are graphs above the horizontal lift γ of $\tilde{\gamma}_0$ with $\gamma(0) = c(0)$. They are generating graphs of the half-cylinder solution. To calculate $I_{\max} = (-R, R)$ of these graphs let us parametrise the graph on $[0, R]$ by arc-length:

Lemma 2.9. *Let $H > H(E)$ and let $h\colon (-R,R) \to \mathbb{R}$ be the symmetric maximal solution with $h(R) = 0$ and $h(0) = a$, established in Lemma 2.6. Parametrising this graph by arc-length and with the same orientation gives a curve $\beta\colon [0,L] \to E$, $\beta = \gamma \circ x + (0,y,0)$ for some $x, y \in \mathcal{C}^2([0,L],\mathbb{R})$ with the following properties:*

- *L is the arc-length of the graph on $[0,R]$, that is $L = \int_0^R \sqrt{1 + h'^2(t)}\, dt$,*
- *β respects the initial values of the graph, i.e., we have $\beta(0) = (0,a,0)$ and $\beta'(0) = \gamma'(0)$,*
- *$\beta(L) = \gamma(R)$ and $\beta'(L) = E_2$.*

For the invariant surface $f\colon \mathbb{R} \times [0,L] \to E$, $f(s,t) := \Phi_s(\beta(t))$ the tangent vectors are

$$v_1 := \frac{\partial f}{\partial s} = \sinh\left(x\sqrt{-\kappa}\right) E_1 + 2\tau \sinh\left(x\sqrt{-\kappa}\right) E_2 + E_3, \tag{2.7}$$

$$v_2 := \frac{\partial f}{\partial t} = x' \operatorname{sech}\left(x\sqrt{-\kappa} t\right) E_1 + y'(t) E_2 - x' \tanh\left(x\sqrt{-\kappa}\right) E_3. \tag{2.8}$$

Proof. The claim about the reparametrisation is clear. For the tangent vector v_1 we have

$$v_1 = K_{\gamma(x(t)) + (0,y(t),0)}.$$

Since the Killing field K is independent of y and γ is the horizontal lift of $\tilde{\gamma}_0$, it suffices to insert

$$x = \frac{\tanh(x(t)\sqrt{-\kappa})}{\sqrt{-\kappa}} \text{ and } z = \frac{\log(\operatorname{sech}(x(t)\sqrt{-\kappa}))}{\sqrt{-\kappa}}$$

into $K_{(x,y,z)}$, given by (2.2), to show (2.7). For (2.8) we note $v_2 = x'\gamma' \circ x + y' E_2$ and refer to (2.4). $\square$

The horizontal diameter of a horizontal MCH-cylinder can be computed using the weight formula; it is independent of τ.

Theorem 2.10. *For the symmetric solution from Lemma 2.6 we have*

$$R = \frac{1}{\sqrt{-\kappa}} \operatorname{arctanh}\left(\frac{\sqrt{-\kappa}}{2H}\right),$$

so that the horizontal diameter *of a horizontal* MCH*-cylinder is* $2R$*. The* MCH*-cylinders with axis* c*, considered as a one-parameter family depending on* $H \in (H(E), \infty)$*, are unbounded for* $H \to H(E) = \frac{\sqrt{-\kappa}}{2}$ *and converge to the horizontal geodesic* c *for the limit* $H \to \infty$*.*

Proof. Let $h\colon (-R, R) \to \mathbb{R}$ be the maximal solution from Lemma 2.6 (e) and let β be the reparametrisation of $h|_{[0,R]}$ by arc-length as in Lemma 2.9. We use the weight formula to determine the explicit value of R. We consider the invariant surface

$$f\colon \mathbb{R} \times [0, L] \to E(\kappa, \tau), \qquad f(s, t) = \Phi_s(\gamma(x(t)) + (0, y(t), 0)).$$

For a bounded domain $\Omega \subset \mathbb{R} \times [0, L]$ with $\partial\Omega$ a closed Jordan curve we let η be the outer unit conormal along $f(\partial\Omega)$ and N is the inner normal of the surface. The weight formula (see [**HdLR05**, Proposition 3] for a proof in a general Riemannian 3-manifold) yields

$$2H \int_{f(\Omega)} \langle N, Y\rangle = \int_{f(\partial\Omega)} \langle \eta, Y\rangle, \qquad Y \text{ Killing field.} \tag{2.9}$$

We apply (2.9) to the Killing field $Y = \xi = E_2$ and set $\Omega := [0, 1] \times [0, L]$.

We need some geometric data of the invariant surface f, which are easily computed with Lemma 2.9:

The entries of the induced metric $g = (\langle v_j, v_k\rangle)_{1\le j,k\le 2}$ on $\mathbb{R} \times J$ are

$$\begin{aligned} g_{11} &= \cosh^2(x\sqrt{-\kappa}) + 4\tau^2 \sinh^2(x\sqrt{-\kappa}),\\ g_{12} &= 2\tau \sinh(x\sqrt{-\kappa}) y'(t),\\ g_{22} &= {x'}^2 + {y'}^2, \end{aligned} \tag{2.10}$$

with

$$\det(g) = \cosh^2(x\sqrt{-\kappa})({x'}^2 + {y'}^2 + 4\tau^2 \tanh^2(x\sqrt{-\kappa}){x'}^2). \tag{2.11}$$

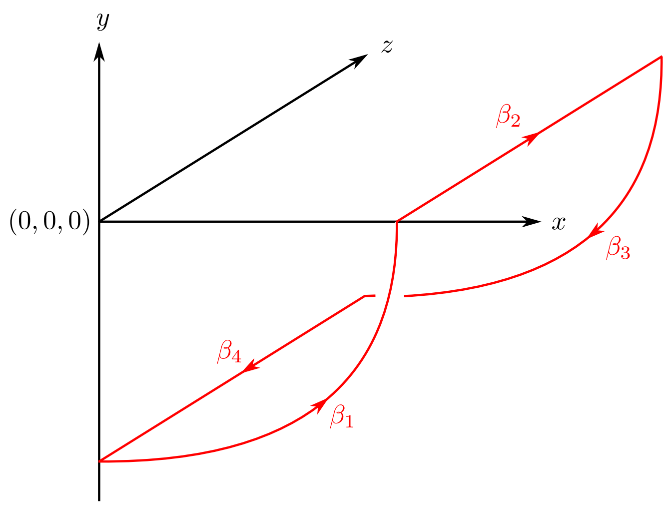


FIGURE 2.1. Application of weight formula

The inner normal N to f satisfies

$$\begin{aligned}\sqrt{\det(g)}N &= \cosh(x\sqrt{-\kappa})y'(-\operatorname{sech}(x\sqrt{-\kappa})E_1 + \tanh(x\sqrt{-\kappa})E_3)\\ &\quad + \cosh(x\sqrt{-\kappa})x'E_2\\ &\quad - 2\tau\sinh(x\sqrt{-\kappa})x'(\tanh(x\sqrt{-\kappa})E_1 + \operatorname{sech}(x\sqrt{-\kappa})E_3).\end{aligned}\tag{2.12}$$

First we compute the left-hand side of (2.9). In view of (2.12) we get

$$2H\int_{f(\Omega)}\langle N, E_2\rangle = 2H\int_{[0,1]\times[0,L]} x'(t)\cosh\left(x(t)\sqrt{-\kappa}\right)ds\,dt = \frac{2H}{\sqrt{-\kappa}}\sinh\left(R\sqrt{-\kappa}\right).$$

To compute the right-hand side of (2.9) we decompose the boundary parametrisation as

$$f(\partial\Omega) = \beta_1 \oplus \beta_2 \oplus \beta_3 \oplus \beta_4,$$

where

$$\beta_1(t) = f(0,t),\quad \beta_2(s) = f(s,L),\quad \beta_3(t) = f(1,L-t),\quad \beta_4(s) = f(1-s,0).$$

See Figure 2.1 above. We denote by η_1 to η_4 the respective unit conormals along β_1 to β_4. Due to $\beta_3(t) = \Phi_1(\beta_1(L-t))$ we have $\beta_3'(t) = -\beta_1'(L-t)$ and thus $\eta_3(t) = -\eta_1(L-t)$.

Since E_2 is a constant Killing field this implies

$$\int_{\beta_1} \langle \eta_1, E_2 \rangle + \int_{\beta_3} \langle \eta_3, E_2 \rangle = 0.$$

To determine $\int_{\beta_4} \langle \eta_4, E_2 \rangle$ note that $\beta_4'(s) = -\frac{\partial f}{\partial s}(1-s,0) = E_3$ and $\frac{\partial f}{\partial t}(1-s,0) = E_1$, i.e., we have $\eta_4 = E_1$. This shows

$$\int_{\beta_4} \langle \eta_4, E_2 \rangle = 0.$$

Finally we consider $\int_{\beta_2} \langle \eta_2, E_3 \rangle$. We note $\beta_2'(s) = v_1$ and for the conormal we get

$$\eta_2 = \frac{1}{\sqrt{g_{11}}\sqrt{\det(g)}} \left(-g_{12} v_1 + g_{11} v_2\right).$$

At L we have

$$x(L) = R, \quad x'(L) = 0, \quad y(L) = 0 \quad \text{and} \quad y'(L) = 1,$$

so that in view of (2.7), (2.8) and (2.10) we get

$$\begin{aligned}\langle \eta_4, E_2 \rangle \sqrt{g_{11}} &= \frac{-4\tau^2 \sinh^2(R\sqrt{-\kappa}) + \cosh^2(R\sqrt{-\kappa}) + 4\tau^2 \sinh^2(R\sqrt{-\kappa})}{\cosh(R\sqrt{-\kappa})} \\ &= \cosh(R\sqrt{-\kappa}).\end{aligned}$$

Noting that $\sqrt{\langle \beta_4', \beta_4' \rangle} = \sqrt{\langle v_1, v_1 \rangle} = \sqrt{g_{11}}$ we get

$$\int_{\beta_4} \langle \eta_4, E_2 \rangle = \int_{[0,1]} \left[\langle \eta_4, E_2 \rangle \cdot \sqrt{\langle \beta_4', \beta_4' \rangle} \right] ds = \cosh(\sqrt{-\kappa} R).$$

Combining these results yields

$$\frac{2H}{\sqrt{-\kappa}} \sinh(R\sqrt{-\kappa}) = \cosh(R\sqrt{-\kappa}).$$

Because of $2H > \sqrt{-\kappa}$ we can solve this equation for R and get

$$R = \frac{1}{\sqrt{-\kappa}} \operatorname{arctanh}\left(\frac{\sqrt{-\kappa}}{2H}\right).$$

The unboundedness for $H \to H(E)$ is clear since $\operatorname{arctanh}(u)$ is unbounded for $u \to 1$. The convergence to c for $H \to \infty$ follows by comparison with MCH-spheres; here we use embeddedness of MCH-cylinders in $E(\kappa, \tau)$-spaces with $\kappa \leq 0$. $\square$

Remark 2.11. We have carried out the same computation for tilted MCH-cylinders in $E(\kappa, 0)$-spaces with $\kappa \leq 0$. The conormals along β_2 and β_4 turn out to be a bit more complicated but as a result we get

$$R = \frac{1}{\sqrt{-\kappa}} \operatorname{arctanh}\left(\frac{\sqrt{-\kappa}}{2H}\right),$$

as in Theorem 2.10. We have not included the computation.

Part 2

Singly periodic constant mean curvature annuli in $\mathbb{H}^2 \times \mathbb{R}$

CHAPTER 3

Preliminaries for Part 2

This chapter addresses various topics since the construction of MCH-surfaces with specific properties is more difficult than the construction of minimal surfaces. A classical method to construct MCH-surfaces with *symmetries* is the *conjugate Plateau construction*. This method is based on the *Lawson correspondence* which says that a simply connected MC1-surface in $\mathbb{R}^3$ is isometric to a minimal surface in the standard 3-sphere. If the MC1-surface has a plane of symmetry then the surface curves in that symmetry plane of $\mathbb{R}^3$ correspond to geodesics in the standard 3-sphere. Thus minimal surfaces bounded by geodesics in $\mathbb{S}^3$ can be used to construct MC1-surfaces in $\mathbb{R}^3$ with specific symmetries. This method has been generalised to the $E(\kappa, \tau)$-spaces and it was used to construct various MCH-surfaces in $\mathbb{H}^2 \times \mathbb{R}$, for example horizontal unduloids via the construction of minimal surfaces in the *Berger spheres*.

Accordingly, we collect the necessary background on the Berger spheres. These represent the compact simply connected homogeneous 3-manifolds with 4-dimensional isometry group. From a geometric point of view they are deformations of the standard 3-sphere $\mathbb{S}^3$ obtained by scaling the Hopf fibres. In particular, the Berger spheres form a one-parameter family of spaces for a given base space.

We then introduce the *Daniel correspondence:* An MCH-surface in the product manifold $\mathbb{H}^2 \times \mathbb{R} = E(-1, 0)$ with $H > \frac{1}{2}$ is isometric to a minimal surface immersed into some Berger sphere. We also include further properties, for example relations of mirror curves in $\mathbb{H}^2 \times \mathbb{R}$ to geodesics in the Berger spheres.

Finally we fix notation for singly periodic surfaces and study symmetries of singly periodic MCH-annuli. In combination with the Daniel correspondence we get an idea what the corresponding minimal surface in the Berger spheres must satisfy. Whenever possible we also highlight the differences to the Lawson correspondence of MC1-surfaces in $\mathbb{R}^3$ and minimal surfaces in $\mathbb{S}^3$.

3.1. Geometry of the Berger spheres

In the following let κ and τ be real numbers with $\kappa > 0$ and $\tau \neq 0$. In this case the simply connected homogeneous 3-manifold $E(\kappa, \tau)$ is compact. We discuss a model and geometric properties.

3.1.1. Model of Berger spheres and isometries. Consider the standard 3-sphere

$$\mathbb{S}^3 := \{p \in \mathbb{R}^4 \colon |p|^2_{\mathbb{R}^4} = 1\}$$

as a set and let V be the matrix

$$V = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

which induces the vector field

$$V(a,b,c,d) = (-b, a, d, -c)$$

on $\mathbb{S}^3$. We endow $\mathbb{S}^3$ with the metric

$$g_{\kappa,\tau}(X,Y) := \frac{4}{\kappa}\left[\langle X, Y\rangle_{\mathbb{R}^4} + \left(\frac{4\tau^2}{\kappa} - 1\right)\langle X, V\rangle_{\mathbb{R}^4}\langle Y, V\rangle_{\mathbb{R}^4}\right]. \tag{3.1}$$

A *Berger sphere* is then the Riemannian space $\mathbb{S}^3(\kappa,\tau) := (\mathbb{S}^3, g_{\kappa,\tau})$.

Remark 3.1. Up to scaling of the metrics the Berger spheres form a one-parameter family of spaces: If we set $\eta := \frac{4\tau^2}{\kappa}$ and scale $g_{\kappa,\tau}$ by $\frac{\kappa}{4}$ the metric depends only on $\eta \in (0,\infty)$.

The isometries of $\mathbb{S}^3(\kappa,\tau)$ can be described as follows:

Proposition 3.2 ([**Tor12**, Section 2]). *We have*

$$\operatorname{Iso}\left(\mathbb{S}^3(\kappa,\tau)\right) = \begin{cases} \mathsf{O}(4) & \text{if } \kappa = 4\tau^2, \\ \{A \in \mathsf{O}(4) \colon AV = \pm VA\} & \text{if } \kappa \neq 4\tau^2. \end{cases} \tag{3.2}$$

3.1.2. Structure of a metric Lie group. The quaternions is the skew-field $\mathbb{R}^4$ considered with basis $\mathbf{1}$, $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ and bilinear product satisfying the relations

$$\mathbf{1}\boldsymbol{i} = \boldsymbol{i}, \quad \boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -\mathbf{1}, \quad \boldsymbol{i}\boldsymbol{j}\boldsymbol{k} = -\mathbf{1}.$$

For $p \in \mathbb{R}^4$ we consider the linear map

$$L_p \colon \mathbb{R}^4 \to \mathbb{R}^4, \qquad L_p(q) := pq.$$

Let $p = a\mathbf{1} + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k}$, then L_p is represented by the matrix

$$A_p = \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix}$$

with respect to the basis $\{\mathbf{1}, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$. For $p \in \mathbb{S}^3$ the matrix A_p is obviously in $\mathsf{O}(4)$ and thus $\mathcal{L}_p := L_p|_{\mathbb{S}^3}$ defines a map $\mathcal{L}_p \colon \mathbb{S}^3 \to \mathbb{S}^3$. Therefore the restriction of this bilinear product induces a group structure on $\mathbb{S}^3$. We have $A_p V = V A_p$ because the 2×2 blocks of A_p and V commute, that is,

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

In view of (3.2) $\mathcal{L}_p$ is an isometry of $\mathbb{S}^3(\kappa, \tau)$ and thus a Berger sphere is a metric Lie group.

3.1.3. Identification with other models. The quaternions have a representation as complex 2×2 matrices since

$$\mathbb{R}^4 \to \mathsf{M}_2(\mathbb{C}), \qquad a\mathbf{1} + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k} \mapsto \begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}$$

is an injective ring homomorphism. The image of $\mathbb{S}^3 \subset \mathbb{R}^4$ is the group $\mathsf{SU}(2)$. Identifying $\mathsf{SU}(2)$ with $\mathbb{S}^3 = \{(z, w) \in \mathbb{C}^2 \colon |z|^2 + |w|^2 = 1\}$ the group structure is as follows:

$$(z_1, w_1)(z_2, w_2) = (z_1 z_2 - w_1 \overline{w}_2, z_1 w_2 + w_1 \overline{z}_2) \quad \text{and} \quad (z, w)^{-1} = (\overline{z}, -w).$$

The neutral element is $(1, 0)$ and left-translations have the following representation:

$$\mathcal{L}_{(z_1, w_1)} \colon \mathbb{S}^3(\kappa, \tau) \to \mathbb{S}^3(\kappa, \tau), \qquad \mathcal{L}_{(z_1, w_1)}(z_2, w_2) := (z_1, w_1)(z_2, w_2).$$

3.1.4. Orthonormal frame and Hopf fibration. At the identity $\mathbf{1} \in \mathbb{S}^3$ the vectors $\boldsymbol{j}$, $\boldsymbol{k}$ and $\boldsymbol{i}$ are orthogonal. Since left-translations are isometries we obtain a global orthonormal frame by setting

$$\begin{aligned} E_1(p) &:= \frac{\sqrt{\kappa}}{2}\mathcal{L}_p(\boldsymbol{j}) = \frac{\sqrt{\kappa}}{2}(-c,-d,a,b) = \frac{\sqrt{\kappa}}{2}(-w,z), \\ E_2(p) &:= \frac{\sqrt{\kappa}}{2}\mathcal{L}_p(\boldsymbol{k}) = \frac{\sqrt{\kappa}}{2}(-d,c,-b,a) = \frac{\sqrt{\kappa}}{2}(iw,iz), \\ \xi(p) &:= \frac{\kappa}{4\tau}\mathcal{L}_p(\boldsymbol{i}) = \frac{\kappa}{4\tau}(-b,a,d-c) = \frac{\kappa}{4\tau}(iz,-iw), \end{aligned} \tag{3.3}$$

where we identify $p = a\mathbf{1}+b\boldsymbol{i}+c\boldsymbol{j}+d\boldsymbol{k} \in \mathbb{R}^4$ with $p = (a+ib, c+id) = (z,w) \in \mathbb{C}^2$.

Using this orthonormal frame and the definition of the Riemannian metric on $\mathbb{S}^3(\kappa,\tau)$ we get the following expression of the Levi-Civita connection:

$$\begin{array}{lll} \nabla_{E_1}E_1 = 0, & \nabla_{E_1}E_2 = \tau\xi, & \nabla_{E_1}\xi = -\tau E_2, \\ \nabla_{E_2}E_1 = -\tau\xi, & \nabla_{E_2}E_2 = 0, & \nabla_{E_2}\xi = \tau E_1, \\ \nabla_{\xi}E_1 = \left(\frac{\kappa}{2\tau} - \tau\right)E_2, & \nabla_{\xi}E_2 = -\left(\frac{\kappa}{2\tau} - \tau\right)E_1, & \nabla_{\xi}\xi = 0. \end{array}$$

We introduce the Hopf fibration and show that this model is an $E(\kappa,\tau)$-space:

Proposition 3.3. *The* Hopf fibration

$$\Pi\colon \mathbb{S}^3(\kappa,\tau) \to \mathbb{S}^2(\kappa), \qquad \Pi(z,w) := \frac{1}{\sqrt{\kappa}}\left(-2izw, |z|^2 - |w|^2\right), \tag{3.4}$$

where $\mathbb{S}^2(\kappa)$ denotes the 2-sphere of radius $\frac{1}{\sqrt{\kappa}}$, is a Riemannian submersion whose fibres are geodesics. The vertical unit Killing vector field is given by ξ. The horizontal space of this submersion is spanned by E_1 and E_2. Moreover, the base space $\mathbb{S}^2(\kappa)$ has constant sectional curvature κ and the bundle curvature of the fibration is τ.

Before we prove this proposition we introduce the following terminology:

Definition. We call ξ *Hopf field*. For $a,b \in \mathbb{R}$ with $a^2+b^2 = 1$ we call the linear combination $F = aE_1 + bE_2$ a *horizontal field*. An integral curve of ξ or F is called a *Hopf circle* or a *(horizontal) F-circle*, respectively.

Proof. Here it is convenient to consider $\mathbb{S}^3 \subset \mathbb{R}^4$. For $p = (a,b,c,d) \in \mathbb{S}^3$ we have

$$\Pi(p) = \frac{1}{\sqrt{\kappa}}\begin{pmatrix} 2(bc+ad) \\ 2(bd-ac) \\ a^2+b^2-c^2-d^2 \end{pmatrix} \quad \text{and} \quad d\Pi_p = \frac{1}{\sqrt{\kappa}}\begin{pmatrix} 2d & 2c & 2b & 2a \\ -2c & 2d & -2a & 2b \\ 2a & 2b & -2c & -2d \end{pmatrix}.$$

We immediately see $\xi(p) \in \ker d\Pi_p$. In order to show that Π is a Riemannian submersion we also have to check that $d\Pi_p$ maps E_1 and E_2 to orthogonal unit vectors in $T_{\Pi(p)}\mathbb{S}^2(\kappa)$. This shows two properties: E_1 and E_2 span the horizontal space (they are orthogonal to the kernel) and $d\Pi_p$ maps the horizontal space isometrically to $T_{\Pi(p)}\mathbb{S}^2(\kappa)$. Noting

$$d\Pi_p E_1(p) = \begin{pmatrix} 2ab - 2cd \\ c^2 - d^2 + b^2 - a^2 \\ -2ac - 2bd \end{pmatrix} \quad \text{and} \quad d\Pi_p E_2(p) = \begin{pmatrix} c^2 - d^2 + a^2 - b^2 \\ 2ab + 2cd \\ 2bc - 2ad \end{pmatrix}$$

we compute

$$\begin{aligned}
\langle d\Pi_p E_1(p), d\Pi_p E_1(p)\rangle_{\mathbb{R}^3} &= \left\langle \begin{pmatrix} 2ab - 2cd \\ c^2 - d^2 + b^2 - a^2 \\ -2ac - 2bd \end{pmatrix}, \begin{pmatrix} 2ab - 2cd \\ c^2 - d^2 + b^2 - a^2 \\ -2ac - 2bd \end{pmatrix} \right\rangle \\
&= 4c^2d^2 + 4a^2b^2 - 8abcd + 4a^2c^2 + 4b^2d^2 + 8abcd \\
&\quad + (c^2 - d^2)^2 + (b^2 - a^2)^2 + 2(c^2 - d^2)(b^2 - a^2) \\
&= c^4 + d^4 + b^4 + a^4 - 2c^2d^2 - 2a^2b^2 + 2c^2b^2 + 2a^2d^2 \\
&\quad - 2b^2d^2 - 2a^2c^2 + 4c^2d^2 + 4a^2b^2 + 4a^2c^2 + 4b^2d^2 \\
&= (c^2 + d^2)^2 + (a^2 + b^2)^2 + 2(a^2 + b^2)(c^2 + d^2) \\
&= \big((a^2 + b^2) + (c^2 + d^2)\big)^2 = 1
\end{aligned}$$

and

$$\begin{aligned}
\langle d\Pi_p E_1(p), d\Pi_p E_2(p)\rangle_{\mathbb{R}^3} &= \left\langle \begin{pmatrix} 2ab - 2cd \\ c^2 - d^2 + b^2 - a^2 \\ -2ac - 2bd \end{pmatrix}, \begin{pmatrix} c^2 - d^2 + a^2 - b^2 \\ 2ab + 2cd \\ 2bc - 2ad \end{pmatrix} \right\rangle \\
&= 4ab(c^2 - d^2) + 4cd(b^2 - a^2) + 4(ad - bc)(ac + bd) = 0.
\end{aligned}$$

The computation for $\langle d\Pi_p E_2(p), d\Pi_p E_2(p)\rangle = 1$ is analogous. Thus Π is a Riemannian submersion and the horizontal space is spanned by E_1 and E_2.

The fibres are integral curves of $\xi = E_3$ and these are geodesics in view of the Levi-Civita connection; we explain this in detail in Subsection 3.1.5.

The base space has obviously constant sectional curvature κ. The bundle curvature is $\frac{1}{2} g_{\kappa,\tau}(\nabla_{E_1} E_2 - \nabla_{E_2} E_1, \xi) = \tau$. Thus $\mathbb{S}^3(\kappa, \tau)$ is an $E(\kappa, \tau)$-space. $\square$

Remark 3.4. For $\kappa = 4\tau^2$ we obtain – up to homotheties – the standard 3-sphere with its usual metric. Whenever this is the case, we will assume $\kappa = 4$ and $\tau = 1$. All tangent vectors are equivalent since the 3-sphere is isotropic (unlike the other Berger spheres). For a unitary imaginary quaternion u, i.e. $u = b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k}$ with $b^2 + c^2 + d^2 = 1$, we also introduce the following terminology:

- The u-*Hopf field* is the left-invariant vector field $p \mapsto pu$ on $\mathbb{S}^3$.
- A u-*Hopf circle* is an integral curve of a u-Hopf field. They are geodesics in $\mathbb{S}^3$.
- The u-*Hopf projection* is the mapping $\Pi_u\colon \mathbb{S}^3 \to \mathbb{S}^2(4)$, $\Pi_u(p) := \frac{1}{2}pup^*$, where $p^* := a\mathbf{1} - b\boldsymbol{i} - c\boldsymbol{j} - d\boldsymbol{k}$. The u-Hopf projection is a Riemannian submersion.

For $\kappa \neq 4\tau^2$ the map $\frac{2}{\sqrt{\kappa}}\Pi_u\colon \mathbb{S}^3(\kappa,\tau) \to \mathbb{S}^2(\kappa)$ is a Riemannian submersion if and only if $u = \pm\boldsymbol{i}$. The Hopf fibration from Proposition 3.3 satisfies $\Pi = \frac{2}{\sqrt{\kappa}}\Pi_{\boldsymbol{i}}$.

3.1.5. Vertical and horizontal geodesics. Let $p = (z, w)$ be in $\mathbb{S}^3(\kappa,\tau)$. Then

$$v(s) := p\cos\left(\frac{\kappa}{4\tau}s\right) + \frac{4\tau}{\kappa}\xi(p)\sin\left(\frac{\kappa}{4\tau}s\right) = \begin{pmatrix} \exp\left(i\frac{\kappa}{4\tau}s\right)z \\ \exp\left(-i\frac{\kappa}{4\tau}s\right)w \end{pmatrix} \tag{3.5}$$

parametrises a vertical unit-speed geodesic through p. Since $\nabla_\xi\xi = 0$, this claim about v follows from $\xi(v(s)) = v'(s)$, that is, v is an integral curve of ξ. It is a fibre since it projects to the point $\Pi(v(s)) = \frac{1}{\sqrt{\kappa}}(-2izw, |z|^2 - |w|^2)$.

Given the horizontal field $F = F_\varphi = \cos(\varphi)E_1 + \sin(\varphi)E_2$, the curve

$$\begin{aligned} h(t) &:= p\cos\left(\frac{\sqrt{\kappa}}{2}t\right) + \frac{2}{\sqrt{\kappa}}F_\varphi(p)\sin\left(\frac{\sqrt{\kappa}}{2}t\right) \\ &= \begin{pmatrix} \cos\left(\frac{\sqrt{\kappa}}{2}t\right)z - \sin\left(\frac{\sqrt{\kappa}}{2}t\right)\exp(-i\varphi)w \\ \cos\left(\frac{\sqrt{\kappa}}{2}t\right)w + \sin\left(\frac{\sqrt{\kappa}}{2}t\right)\exp(i\varphi)z \end{pmatrix} \end{aligned} \tag{3.6}$$

parametrises a horizontal unit-speed geodesic through p with tangent vector F_φ. This is a consequence of $\nabla_{F_\varphi}F_\varphi = 0$ and $F_\varphi(h(t)) = h'(t)$.

We read off the lengths of the Hopf circle v and of the horizontal F-circle h.

Observation. *Vertical and horizontal geodesics have the following respective lengths:*

$$\operatorname{length}(v) = \frac{8\tau\pi}{\kappa} \quad \textit{and} \quad \operatorname{length}(h) = \frac{4\pi}{\sqrt{\kappa}}. \tag{3.7}$$

3.1.6. An important minimal surface: the spherical helicoid. Let h_1 and h_2 be horizontal geodesics in $\mathbb{S}^3(\kappa,\tau)$. It is natural to ask the following question: Is there a minimal surface bounded by h_1 and h_2? In Chapter 5 we will answer the question in general for two disjoint Jordan curves in a compact Riemannian 3-manifold.

In the next proposition we will show that in the special case of the Berger spheres we always have an explicit solution as long as h_1 and h_2 are either identical or linked. If they are linked then they bound a *spherical helicoid* with a vertical axis v joining h_1 and h_2; the rulings are horizontal geodesics which rotate (with constant angular speed) about the axis. Letting the pitch of such a helicoid go to 0 we arrive at the case that h_1 and h_2 are identical. Then $h_1 = h_2$ bounds a so-called *horizontal umbrella*.

For the upcoming chapters it is useful to describe these helicoids explicitly:

Proposition 3.5. *Let h_1 and h_2 be identical or linked horizontal geodesics in $\mathbb{S}^3(\kappa,\tau)$. Then, up to a left-translation, a minimal surface bounded by h_1 and h_2 is given by*

$$f\colon \mathbb{R}\times[0,1]\to\mathbb{S}^3(\kappa,\tau),\qquad f(x,y):=\begin{pmatrix}\cos\left(\frac{\sqrt{\kappa}}{2}x\right)\exp\left(\pm i\frac{\kappa}{4\tau}\ell y\right)\\ \sin\left(\frac{\sqrt{\kappa}}{2}x\right)\exp\left(i\left(\varphi\pm\frac{\kappa}{4\tau}\ell\right)y\right)\end{pmatrix},$$

where $\ell\in[0,8\tau\pi/\kappa)$ and $\varphi\in[0,2\pi]$. The Hopf projections $\Pi\circ h_1$ and $\Pi\circ h_2$ enclose an angle of $\varphi\pm\frac{\kappa}{2\tau}\ell$.

Proof. The curves $\Pi\circ h_1$ and $\Pi\circ h_2$ are geodesics in $\mathbb{S}^2(\kappa)$ and thus they intersect in at least two points. Therefore h_1 and h_2 are joined by a segment of a vertical geodesic v. Since h_1 and h_2 are assumed to be either identical or linked, any such segment has length $\ell\in[0,8\tau\pi/\kappa)$. After a left-translation we can assume that h_1 is a horizontal geodesic through $p=(1,0)$ with $u=E_1$ and that v starts from $h_1(0)=(1,0)$, i.e., $\Pi\circ h_1$ and $\Pi\circ h_2$ intersect at the point $\frac{1}{\sqrt{\kappa}}(0,0,1)$.

By (3.6) and (3.5) the curves h_1 and v are parametrised as follows:

$$h_1\colon[0,4\pi/\sqrt{\kappa}]\to\mathbb{S}^3(\kappa,\tau),\qquad h_1(t):=\begin{pmatrix}\cos\left(\frac{\sqrt{\kappa}}{2}t\right)\\ \sin\left(\frac{\sqrt{\kappa}}{2}t\right)\end{pmatrix}$$

and

$$v\colon[0,\ell]\to\mathbb{S}^3(\kappa,\tau),\qquad v(s):=\begin{pmatrix}\exp\left(\pm i\frac{\kappa}{4\tau}s\right)\\ 0\end{pmatrix}.$$

Consequently the left-translation of h_2 goes through $v(\ell)$ and as horizontal field of h_2 we assume $F_\varphi = \cos(\varphi)E_1 + \sin(\varphi)E_2$ for some $\varphi \in [0, 2\pi]$. Then (3.6) yields

$$h_2\colon [0, 4\pi/\sqrt{\kappa}] \to \mathbb{S}^3(\kappa,\tau), \qquad h_2(t) = \begin{pmatrix} \cos\left(\frac{\sqrt{\kappa}}{2}t\right)\exp\left(\pm i\frac{\kappa}{4\tau}\ell\right) \\ \sin\left(\frac{\sqrt{\kappa}}{2}t\right)\exp\left(i\left(\varphi \pm \frac{\kappa}{4\tau}\ell\right)\right) \end{pmatrix}.$$

For $y \in \mathbb{R}$ we consider

$$\psi_y\colon \mathbb{S}^3(\kappa,\tau) \to \mathbb{S}^3(\kappa,\tau), \qquad \psi_y(z,w) := \begin{pmatrix} z\exp\left(\pm i\frac{\kappa}{4\tau}\ell y\right) \\ w\exp\left(i\left(\varphi \pm \frac{\kappa}{4\tau}\ell\right)y\right) \end{pmatrix}.$$

We claim that each ψ_y is an isometry. Left-translation along v is given by

$$(z,w) \mapsto \begin{pmatrix} z\exp\left(\pm i\frac{\kappa}{4\tau}\ell y\right) \\ w\exp\left(\pm i\frac{\kappa}{4\tau}\ell y\right) \end{pmatrix}$$

and the map

$$(z,w) \mapsto \begin{pmatrix} z \\ w\exp\left(i\varphi y\right) \end{pmatrix}$$

is a rotation about the fibre v. Both of these maps are isometries, easily verified in view of (3.2). Hence the composition ψ_y is an isometry, too. We have $\psi_y \circ \psi_z = \psi_{y+z}$ for all $y, z \in \mathbb{R}$ and $\psi_0 = \mathrm{id}$, that is, $(\psi_y)_{y\in\mathbb{R}}$ is a one-parameter group of isometries.

We consider f on the domain $\mathbb{R}^2$ and in view of $f(x,y) = \psi_y(h_1(x))$ it is sufficient to show that the mean curvature vanishes along $h_1(x) = f(x,0)$. To do so we introduce $\rho(z,w) := (\overline{z}, \overline{w})$, the rotation of angle π about h_1. It is in isometry and due to $\rho(f(x,y)) = f(x,-y)$ the surface is invariant under ρ. On the one hand the mean curvature remains the same when applying ρ, on the other hand the normal changes its sign and so does the mean curvature H. We conclude $H \equiv 0$ along h_1. $\square$

3.2. Daniel and Lawson correspondence

We introduce the *Daniel correspondence* by Daniel from [**Dan07**] and its properties à la Große-Brauckmann and Kusner in [**GBK12**, Section 2] respectively Manzano and Torralbo in [**MT14**]. We refer to [**GB05**, Section 2 and Section 3] for the *Lawson correspondence*. Most results we state in this section are quotations from these papers.

Fist we introduce some notation:

Definition. Let Σ be an oriented surface immersed into some Riemannian manifold E. Then the Riemannian metric $\langle \cdot, \cdot \rangle$ of E induces a rotation of angle $\frac{\pi}{2}$ on the tangent bundle to Σ. We denote this rotation by J. The Levi-Civita connection ∇ of E defines the *shape operator* S by $SX := -\nabla_X N$ where X is tangent to Σ and N is a unit normal vector field on Σ.

Recall that for $\kappa, \tau \in \mathbb{R}$ we denote by $E(\kappa, \tau)$ the simply connected homogeneous 3-manifold with base curvature κ and bundle curvature τ. Constructing MCH-surfaces with $H \neq 0$ and boundary in vertical or horizontal planes of a product manifold is difficult, because it is a free boundary problem. The Daniel correspondence reduces this free boundary problem to a fixed boundary problem, but one has to deal with more complicated ambient spaces:

Proposition 3.6 (special case of [**Dan07**, Theorem 5.2])**.** *Let κ and H be in $\mathbb{R}$. For each simply connected*

$$\textit{minimal surface } \Sigma \subset E(\kappa + 4H^2, H) \textit{ with shape operator } S$$

there exists an isometric

$$\textit{MC}H\textit{-surface } \widetilde{\Sigma} \subset E(\kappa, 0) = M(\kappa) \times \mathbb{R} \textit{ with shape operator } \widetilde{S} = JS + H\,\mathrm{id} \tag{3.8}$$

and vice versa.

We will refer to the surfaces Σ and $\widetilde{\Sigma}$ as *conjugate sister surfaces*, or simply as *sisters*. Let us take a look at some special choices for κ and H:

Example 3.7. (a) For $\kappa = 0$ and $H = 1$ we obtain Lawson's correspondence of minimal surfaces in $\mathbb{S}^3$ and MC1-surfaces in $\mathbb{R}^3$.

(b) MCH-surfaces $\widetilde{\Sigma}$ in the space $E(-1, 0) = \mathbb{H}^2 \times \mathbb{R}$ with $H > \frac{1}{2}$ have minimal sisters Σ in Berger spheres $E(4H^2 - 1, H)$ since the base curvature $4H^2 - 1$ is positive.

Daniel's correspondence has the following *first order description*:

Proposition 3.8 (part of [**Dan07**, Theorem 5.2])**.** *We suppose that a minimal immersion $f\colon \Omega \to E(4H^2 + \kappa, H)$ and an MCH-immersion $\tilde{f}\colon \Omega \to E(\kappa, 0)$ parametrise sister surfaces with unit normal fields N and $\widetilde{N}$, respectively. Let Z and $\widetilde{Z}$ be the corresponding* *restrictions of the vertical vector fields ξ and $\widetilde{\xi}$ along f and $\tilde{f}$, respectively. Denote the projections to the respective tangents spaces of these vector fields by T and $\widetilde{T}$. Then*

$$\big\langle \widetilde{Z}, \widetilde{N} \big\rangle = \langle Z, N \rangle \quad \text{and} \quad df^{-1}(T) = J d\tilde{f}^{-1}\big(\widetilde{T}\big). \tag{3.9}$$

In an $E(\kappa, \tau)$-space, a rotation of angle π about a horizontal or vertical geodesic is an isometry. In the product spaces $E(\kappa, 0) = M(\kappa) \times \mathbb{R}$, reflections in vertical and horizontal planes are isometries. We refer to these planes as *mirror planes*. A curve $\tilde{c}$ on a surface $\tilde{\Sigma}$ in $E(\kappa, 0)$ is called *mirror curve* if it is contained in a mirror plane and its conormal $\tilde{\eta}$ is perpendicular to the mirror plane.

The correspondence relates mirror curves as follows:

Proposition 3.9 ([**GBK12**, Proposition 2.5])**.** *Let $\widetilde{\Sigma}$ be an* MC*H-surface in $M(\kappa) \times \mathbb{R}$ with sister minimal surface Σ in $E(4H^2 + \kappa, H)$.*

(i) A curve $\tilde{c}$ on $\widetilde{\Sigma} \subset M(\kappa) \times \mathbb{R}$ is a vertical mirror curve if and only if its sister curve c on $\Sigma \subset E(\kappa + 4H^2, H)$ is a horizontal (ambient) geodesic.

(ii) Similarly, $\tilde{c}$ is contained in a horizontal mirror curve if and only if c on the minimal sister $\Sigma \subset E(4H^2 + \kappa, H)$ is a vertical geodesic (contained in a fibre).

Another issue is the smooth extension of surfaces bounded by geodesics. If a minimal surface Σ in some $E(\kappa, \tau)$-space has a vertical or horizontal geodesic c contained in $\partial\Sigma$ then it is possible to extend Σ by geodesic reflection around c. This is better known as *Schwarz reflection* and the extension is smooth; for details we refer to [**MT14**, Section 2.2] or [**GBK12**, Section 4.2]. The Daniel correspondence also relates these extensions:

Proposition 3.10 ([**MT14**, Lemma 2])**.** *Let an* MC*H-immersion $\tilde{f} \colon \Omega \to E(\kappa, 0)$ and a minimal immersion $f \colon \Omega \to E(4H^2 + \kappa, H)$ parametrise sister surfaces. If $f(\Omega)$ is invariant by a $\left\{ \begin{matrix} \textit{horizontal} \\ \textit{vertical} \end{matrix} \right\}$ geodesic reflection, then $\tilde{f}(\Omega)$ is invariant by a reflection in a $\left\{ \begin{matrix} \textit{vertical} \\ \textit{horizontal} \end{matrix} \right\}$ plane.*

We now address the case $\kappa = 4$ and $\tau = 1$. We choose the coordinate system in $\mathbb{R}^3$ as pictured in Figure 3.1 and think of $\boldsymbol{i}$ as a vertical direction. A unit quaternion $u = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$ can thus be regarded as a direction in $\mathbb{R}^3$. We obtain the following relation for mirror curves:

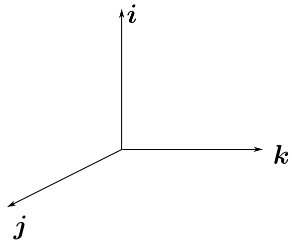


FIGURE 3.1. Coordinates in $\mathbb{R}^3$

Proposition 3.11 ([**GB05**, Corollary 3.1]). *Let $\widetilde{\Sigma}$ be an* MC1*-surface in $\mathbb{R}^3$ and Σ its minimal sister in $\mathbb{S}^3$. Then $\tilde{c}$ on $\widetilde{\Sigma}$ is a curve of planar reflection, for a plane perpendicular to $u \in \mathbb{S}^2$, if and only if the sister curve c on Σ traces out a u-Hopf circle of $\mathbb{S}^3$.*

Remark 3.12. Comparing Proposition 3.9 and Proposition 3.11 a natural question arises: Let $\tilde{c}$ be a vertical mirror curve of an MCH-surface $\widetilde{\Sigma} \subset \mathbb{H}^2 \times \mathbb{R}$ with $H > \frac{1}{2}$. Is the sister curve c on $\Sigma \subset \mathbb{S}^3(4H^2 - 1, H)$ the integral curve of a fixed horizontal field F?

In Theorem 4.1 (iii) we show that the two vertical mirror curves of one half of a vertical unduloid in $\mathbb{H}^2 \times \mathbb{R}$ correspond to a F_1-circle and F_2-circle with $F_1 \neq \pm F_2$. In this sense the statement of Proposition 3.11 does not hold for the Daniel correspondence. The reason is that the Lawson correspondence of MC1-surfaces in $\mathbb{R}^3$ and minimal surfaces in $\mathbb{S}^3$ admits a first order description for any Killing field, not only for the vertical Killing field as it is the case in the Daniel correspondence.

3.3. Singly periodic surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{R}^3$: Basic definitions and properties

We define singly periodic surfaces in the product spaces $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{R}^3$ as surfaces invariant under specific (discrete) groups of isometries. We then obtain symmetries of singly periodic (Alexandrov) embedded MCH-annuli.

In the paragraph preceding Proposition 2.2 we discussed the translations $(\Phi_s)_{s\in\mathbb{R}}$ induced by a geodesic $\tilde{\gamma}$ with geodesic projection. In case of $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{R}^3$ any geodesic has a geodesic projection. We also note that different geodesics $\tilde{\gamma}$ can induce the same one-parameter family $(\Phi_s)_{s\in\mathbb{R}}$. For instance, translations along any vertical geodesic in $\mathbb{H}^2 \times \mathbb{R}$ or translations along any parallel geodesics in $\mathbb{R}^3$.

Definition (Singly periodic)**.** A surface $\widetilde{\Sigma}$ in $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{R}^3$ is called *singly periodic* if the following is satisfied: There is a one-parameter family of isometries $(\Phi_s)_{s\in\mathbb{R}}$, induced by translations along a geodesic $\tilde{\gamma}$ as in the paragraph preceding Proposition 2.2, and a real number $T > 0$ such that $\widetilde{\Sigma}$ is invariant under the discrete group $\{\Phi_{nT} \colon n \in \mathbb{Z}\}$. We call any geodesic $\tilde{\gamma}$ inducing the translations $(\Phi_s)_{s\in\mathbb{R}}$ an *axis* of $\widetilde{\Sigma}$.

Two geodesics $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ in $\mathbb{R}^3$ generate $(\Phi_s)_{s\in\mathbb{R}}$ if and only if $\tilde{\gamma}_1' = \tilde{\gamma}_2'$. The same question in $\mathbb{H}^2 \times \mathbb{R}$ has a more interesting answer, which we highlight along another important property of singly periodic surfaces:

Proposition 3.13. *Let $\widetilde{\Sigma}$ be a singly periodic surface invariant under $(\Phi_{nT})_{n\in\mathbb{Z}}$.*

(i) Suppose $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ both generate $(\Phi_s)_{s\in\mathbb{R}}$ in $\mathbb{H}^2 \times \mathbb{R}$. Then either both of them are vertical geodesics or they lie in the same vertical plane and differ by a vertical translation.

(ii) Let $\widetilde{\Sigma}$ be a singly periodic immersed annulus in $\mathbb{R}^3$ or $\mathbb{H}^2 \times \mathbb{R}$. Then there exists an annulus $\widetilde{\Sigma}_0 \subset \widetilde{\Sigma}$ and $m \in \mathbb{N}$ such that $\widetilde{\Sigma} = \Big\{\Phi_{nmT}(\widetilde{\Sigma}_0) \colon n \in \mathbb{Z}\Big\}$. In particular $\widetilde{\Sigma}$ is a proper immersion.

Proof. For (i) we distinguish two cases for $(\Phi_s)_{s\in\mathbb{R}}$. If $\tilde{\gamma}_1$ is a vertical geodesic, then $\Phi_s(x, y, z) = (x, y, z+s)$ and $\tilde{\gamma}_2(s) = \Phi_s(\tilde{\gamma}_2(0))$ is a vertical geodesic, too. If $\tilde{\gamma}_1$ is non-vertical then Φ_s is the composition of a hyperbolic translation along a horizontal geodesic and a vertical translation. A hyperbolic isometry in $\mathbb{H}^2$ fixes exactly one geodesic and thus $(\Phi_s)_{s\in\mathbb{R}}$ fixes exactly one vertical plane. The geodesic $\tilde{\gamma}_2$ induces $(\Phi_s)_{s\in\mathbb{R}}$ as well and so it must lie in the same vertical plane. The geodesics $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are orbits of $(\Phi_s)_{s\in\mathbb{R}}$ and thus they have the same slope with respect to the vertical Killing field $\widetilde{\xi}$ of $\mathbb{H}^2 \times \mathbb{R}$. Therefore they differ by a vertical translation.

For (ii) let $\Omega := \{(x, y) \in \mathbb{R}^2 \colon 0 < x^2 + y^2 < 1\}$ and $\tilde{f} \colon \Omega \to E(\kappa, 0)$ be an immersion such that $\widetilde{\Sigma} = \tilde{f}(\Omega)$. Let $\alpha \colon [0, 2\pi), \alpha(t) := \frac{1}{2}(\cos(t), \sin(t))$. Then $\tilde{f} \circ \alpha$ has compact image and for $m \in \mathbb{N}$ sufficiently large $\tilde{f} \circ \alpha$ and $\Phi_{mT} \circ \tilde{f} \circ \alpha$ are disjoint due to the compactness of $\tilde{f} \circ \alpha$. Thus we find a curve β in Ω that is disjoint from α and satisfies $\Phi_{mT} \circ \tilde{f} \circ \alpha = \tilde{f} \circ \beta$. As illustrated in Figure 3.2, α and β bound an annulus Ω_0 in Ω. The image $\tilde{\Sigma}_0 := \tilde{f}(\Omega_0)$ is an annulus with the desired properties. $\square$

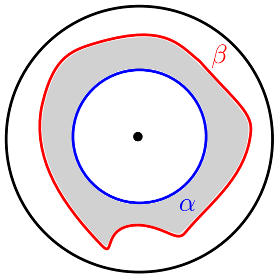


FIGURE 3.2. Shaded in grey is the annulus Ω_0 bounded by α and β

In the following we want to apply Alexandrov reflection. For this purpose we recall the notion of Alexandrov embeddedness:

Definition. Let M be a Riemannian 3-manifold. A surface $\widetilde{\Sigma}$ in M is said to be of *finite topology* if it is a proper immersion of $\mathbb{S}^2 \setminus \{q_1, \ldots, q_k\}$. A surface $\widetilde{\Sigma}$ of finite topology is *Alexandrov embedded* in M if there exists a compact Riemannian 3-manifold W with boundary $\partial W = \mathbb{S}^2$ and a proper immersion $F\colon W \setminus \{q_1, \ldots, q_k\} \to M$ whose boundary restriction $f\colon \mathbb{S}^2 \setminus \{q_1, \ldots, q_k\}$ parametrises $\widetilde{\Sigma}$.

The case of a properly immersed annulus corresponds to $k = 2$, that is, a properly immersed two-punctured sphere. We study symmetries of singly periodic (Alexandrov) embedded MCH-annuli in the product $\mathbb{H}^2 \times \mathbb{R}$; to point to the analogy, we also include the case of $\mathbb{R}^3$. The main property is that such surfaces always have a vertical mirror plane:

Proposition 3.14. *Let $\widetilde{\Sigma}$ be a singly periodic (Alexandrov) embedded MCH-annulus with axis $\tilde{\gamma}$ in $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{R}^3$.*

(i) If $\widetilde{\Sigma} \subset \mathbb{R}^3$ and $H > 0$ then $\widetilde{\Sigma}$ is rotationally invariant, i.e. an unduloid; in particular, it has a vertical mirror plane P such that P separates $\widetilde{\Sigma}$ into simply connected MCH-surfaces $\widetilde{\Sigma}_\pm$ with $\partial\widetilde{\Sigma}_\pm \subset P$ and $\widetilde{\Sigma} = \widetilde{\Sigma}_+ \cup \widetilde{\Sigma}_-$.

(ii) In case $\widetilde{\Sigma} \subset \mathbb{H}^2 \times \mathbb{R}$ and $H > \frac{1}{2}$ there exists a vertical mirror plane as in (i). Moreover, if $\tilde{\gamma}$ is vertical then $\widetilde{\Sigma}$ is invariant under rotations about $\tilde{\gamma}$. If $\tilde{\gamma}$ is horizontal then the horizontal and the vertical plane containing $\tilde{\gamma}$ are mirror planes of $\widetilde{\Sigma}$.

Proof. (i): Let P be a plane containing $\tilde{\gamma}$ and $(P_s)_{s \in \mathbb{R}}$ be the family of planes parallel to P such that $\operatorname{dist}(P_s, P) = s$ for all $s \in \mathbb{R}$. There are $a, b \in \mathbb{R}$ with $a < b$ such that $\widetilde{\Sigma}$ is contained in the slab $S_{[a,b]} = \bigcup_{s\in[a,b]} P_s$ since $\widetilde{\Sigma}$ is generated by a compact annulus. We can assume that P_a and P_b are disjoint from $\widetilde{\Sigma}$. As we let s go from a to b the planes P_s will, at some point, intersect $\widetilde{\Sigma}$. We apply Alexandrov reflection with respect to $(P_s)_{s\in[a,b]}$ and obtain a tangential contact in the moving planes argument since $\widetilde{\Sigma}$ is proper (boundary is at infinity) and generated by a compact annulus (this excludes tangential contact at infinity). Therefore some P_s is a mirror plane. As in [**KKS89**] we conclude that $\widetilde{\Sigma}$ is rotationally invariant. The claim about the vertical mirror plane follows since there is a vertical plane containing $\tilde{\gamma}$.

(ii): Here we only have vertical and horizontal planes at hand. For a vertical axis $\tilde{\gamma}$ we can reason as in (i). If $\tilde{\gamma}$ is non-vertical then its projection is a geodesic of $\mathbb{H}^2$. Let $\tilde{c}$ be a geodesic in $\mathbb{H}^2$ which intersects the projection of $\tilde{\gamma}$ orthogonally. The family of geodesics orthogonal to $\tilde{c}$ defines a family of vertical planes $(P_s)_{s\in\mathbb{R}}$ such that $\tilde{\gamma}$ is in P_0. Alexandrov's moving planes argument shows that P_0 is a mirror plane of $\widetilde{\Sigma}$. If $\tilde{\gamma}$ is horizontal then we can reason in the same way with respect to horizontal planes. □

Remark 3.15. Consider a properly (Alexandrov) embedded MCH-annulus in $\mathbb{R}^3$ which is not assumed to be singly periodic. Then the problem arises that the first tangential contact in the moving planes argument possibly occurs at infinity. In [**KKS89**] an argument involving tilted planes was used to overcome this difficulty for MCH-annuli in $\mathbb{R}^3$ and this way they were shown to be rotationally invariant. This argument is not available in $\mathbb{H}^2 \times \mathbb{R}$. Nevertheless, Mazet showed in [**Maz15**] that a properly (Alexandrov) embedded MCH-annulus contained in a vertical cylinder must be rotationally invariant. He uses and verifies specific properties of the Alexandrov function associated to a family of moving planes by an analytical approach.

Daniel's correspondence immediately implies the following:

Corollary 3.16. *Let $\widetilde{\Sigma} \subset \mathbb{H}^2 \times \mathbb{R}$ be a singly periodic (Alexandrov) embedded MCH-annulus with $H > \frac{1}{2}$. Then the minimal sister surface $\Sigma_\pm \subset \mathbb{S}^3(4H^2-1, H)$ of $\widetilde{\Sigma}_\pm$ is bounded by horizontal geodesics h_1 and h_2.*

CHAPTER 4

Revisiting known examples

We recall that our ultimate goal is to construct *tilted unduloids* in $\mathbb{H}^2 \times \mathbb{R}$, that is, singly periodic (Alexandrov) embedded MCH-annuli with $H > \frac{1}{2}$ with neither vertical nor horizontal axis. To construct such a surface we want to start with a minimal surface in $\mathbb{S}^3(4H^2 - 1, H)$. Corollary 3.16 is a necessary condition for what kind of minimal surfaces we have to look for in $\mathbb{S}^3(4H^2 - 1, H)$, namely, for minimal surfaces bounded by horizontal geodesics. For such a *conjugate Plateau construction* it is important to prescribe the correct boundary contour and topology of the desired minimal surfaces in order to control the geometry of the corresponding MCH-surface in $\mathbb{H}^2 \times \mathbb{R}$. Therefore we want to revisit the known examples of singly periodic (Alexandrov) embedded MCH-annuli with $H > \frac{1}{2}$ in $\mathbb{H}^2 \times \mathbb{R}$. These examples are vertical unduloids, horizontal unduloids constructed by Manzano and Torralbo and the tilted MCH-cylinders from Theorem 2.7. Any such surface $\widetilde{\Sigma}$ has a vertical mirror plane P that separates it into two simply connected MCH-surfaces $\widetilde{\Sigma}_\pm$ with $\partial\widetilde{\Sigma}_\pm \subset P$, as we have shown in the previous chapter. We observe that the corresponding minimal surface $\Sigma_\pm$ in $\mathbb{S}^3(4H^2 - 1, H)$ is a minimal annulus bounded by linked horizontal geodesics h_1 and h_2. Moreover, for one half of a vertical and a horizontal unduloid, we determine the horizontal fields F_1 and F_2 of h_1 and h_2 explicitly. With this information at hand we can formulate a problem in $\mathbb{S}^3(4H^2 - 1, H)$ whose solution we expect to correspond to tilted unduloids in $\mathbb{H}^2 \times \mathbb{R}$. To complement our picture we also include the case of unduloids in $\mathbb{R}^3$ and their minimal sisters in $\mathbb{S}^3$.

4.1. Vertical and horizontal unduloids in $\mathbb{H}^2 \times \mathbb{R}$ and their sisters

The existence of vertical unduloids in $\mathbb{H}^2 \times \mathbb{R}$ as MCH-surfaces of revolution with $H > \frac{1}{2}$ about the fibre has been established established by Wu-Teh Hsiang and Wu-Yi Hsiang in [**HH89**]. In [**MT14**] it is shown that a spherical helicoid, considered as a surface in $\mathbb{S}^3(4H^2 - 1, H)$, is the sister surface of such a surface of revolution. We study the spherical helicoid in $\mathbb{S}^3(4H^2-1,H)$ in somewhat more detail, namely we compute the shape operator of it, in order to determine which part of the spherical helicoid corresponds to one half of a vertical unduloid in $\mathbb{H}^2\times\mathbb{R}$. It turns out that for a vertical unduloid the boundary curves of the sister surface are horizontal circles whose horizontal fields are linearly independent. However, for a horizontal unduloid, constructed as in [**MT14**], we show that the boundary curves in the vertical mirror plane correspond to horizontal geodesics with the same horizontal field up to a sign. We finish this section with the observation that the sister curves of the horizontal unduloid bound at least two solutions; one solution corresponds to the horizontal unduloid and the other one to a piece of a vertical unduloid.

The first example is the vertical unduloid. It arises as the sister surface of the spherical helicoid from Proposition 3.5 if we choose $\ell\in\left[0,\frac{2\tau}{\kappa}\varphi\right]$. The parameter $c:=\frac{\ell}{\frac{4\tau}{\kappa}-\ell}$ then satisfies $0\le c\le 1$, allowing us to consider the reparametrisation f^c defined in Theorem 4.1. The parameter c describes *neck size* of the vertical unduloid, with the extremal cases $c=1$ and $c=0$ corresponding to a vertical cylinder and a sphere, respectively.

Theorem 4.1 (for (ii) compare with [**MT14**, Proposition 1])**.** *Let $c\in[0,1]$ and consider the (reparametrised) spherical helicoid*

$$f^c\colon\mathbb{R}^2\to\mathbb{S}^3(\kappa,\tau),\qquad f^c(x,y):=\begin{pmatrix}\cos\left(\frac{\sqrt{\kappa}}{2}x\right)\exp\left(-ic\frac{\kappa}{4\tau}y\right)\\ \sin\left(\frac{\sqrt{\kappa}}{2}x\right)\exp\left(i\frac{\kappa}{4\tau}y\right)\end{pmatrix}.$$

It is an immersion on $\mathbb{R}^2$ for all $c\in(0,1]$, respectively on $(0,\pi/\sqrt{\kappa})\times\mathbb{R}$ for $c=0$. It has the following properties:

(i) For all $\kappa>0$ and $\tau\in\mathbb{R}$ the spherical helicoid is a minimal surface and the curve $v:=f^c(\pi/\sqrt{\kappa},\cdot)$ is a vertical geodesic on f^c. Each meridian $x\mapsto f(x,y)$ is a horizontal F-circle with

$$F=\frac{\partial f}{\partial x}(x,y)=\cos\Big((c+1)\frac{\kappa}{4\tau}y\Big)E_1+\sin\Big((c+1)\frac{\kappa}{4\tau}y\Big)E_2.$$

(ii) For $H>\frac{1}{2}$ the sister curve $\widetilde{v}$ of v in $\mathbb{S}^3(4H^2-1,H)$ is a curve of constant curvature $k_{\widetilde{v}}=2H+(c-1)\frac{4H^2-1}{4H}>1$ in a horizontal plane of $\mathbb{H}^2\times\mathbb{R}$. The sister surface of f^c is a surface of revolution with constant mean curvature H.

(iii) The curves $h_1:=f^c(\cdot,0)$ and $h_2:=f^c(\cdot,T)$, where

$$T=\frac{\pi}{\sqrt{k_{\widetilde{v}}^2-1}}>0,$$

bound the sister surface corresponding to one half of the vertical unduloid $\widetilde{f^c}$. For $c \in (0,1]$ the respective horizontal fields F_1 and F_2 are linearly independent, that is, they satisfy $F_1 \neq \pm F_2$.

Remark 4.2. We note that f^c has also been studied in [**MT14**, Proposition 1]. They sketch the arguments needed to show that the MCH-sister surface of f^c is rotationally invariant, but they do not determine the piece of f^c corresponding to one half of a vertical unduloid.

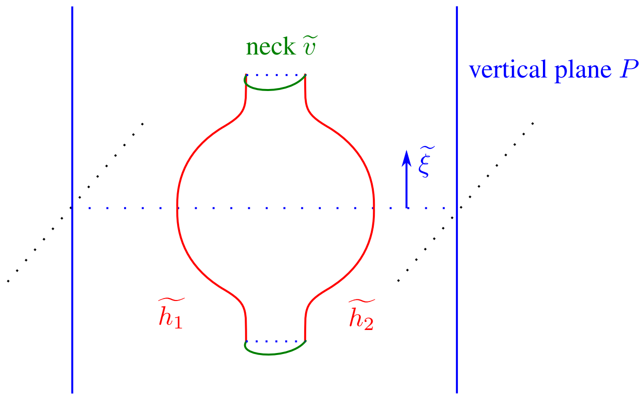


FIGURE 4.1. Sister surface of f^c in $\mathbb{H}^2 \times \mathbb{R}$; h_1 and h_2 are chosen as in Theorem 4.1 (iii) and the neck is contained in a horizontal plane with normal $\widetilde{\xi}$

The extremal cases for c are instructive and useful for the proof of (iii), so that we first consider an example before proceeding with the proof of Theorem 4.1.

Example 4.3 (Vertical cylinder and sphere)**.** The extremal cases are $c = 1$ and $c = 0$.

(a) The vertical cylinder corresponds to $c = 1$. In that case we have $k_{\widetilde{v}} = 2H$ and thus $T = \frac{\pi}{\sqrt{4H^2-1}}$. By Theorem 4.1 (i) and (iii) we have $h_1' = E_1$ and

$$h_2' = \cos\left(\frac{\sqrt{4H^2-1}}{2H}\pi\right) E_1 + \sin\left(\frac{\sqrt{4H^2-1}}{2H}\pi\right) E_2.$$

We have $\frac{\sqrt{4H^2-1}}{2H} \in (0,1)$ for $H > \frac{1}{2}$ so that $F_2 \neq -F_1$.

(b) The MCH-sphere corresponds to $c = 0$. We then have $k_{\widetilde{v}} = 2H - \frac{4H^2-1}{4H} = \frac{4H^2+1}{4H}$ and consequently

$$T = \frac{\pi}{\sqrt{\left(\frac{4H^2+1}{4H}\right)^2 - 1}} = \pi \cdot \frac{4H}{4H^2-1}.$$

This shows $F_1 = -F_2$. In fact, h_1 and h_2 are part of the same horizontal geodesic, just with opposite orientations.

Proof. For (i) we argue as in Proposition 3.5 to see that f^c is a minimal surface. It is also clear that v is a vertical geodesic since its Hopf projection is a point. The claim about the meridians and the horizontal field F follows from a look at (3.6).

(ii): Let $v_1 := \frac{\partial f^c}{\partial x}(x, y)$ and $v_2 := \frac{\partial f^c}{\partial y}(x, y)$. At $x = \pi/\sqrt{\kappa}$ we have

$$v_1 = \cos\Big((c+1)\frac{\kappa}{4\tau}y\Big)\, E_1 + \sin\Big((c+1)\frac{\kappa}{4\tau}y\Big)\, E_2 \quad \text{and} \quad v_2 = -\xi.$$

Thus $N = -\sin\big((c+1)\frac{\kappa}{4\tau}y\big)\, E_1 + \cos\big((c+1)\frac{\kappa}{4\tau}y\big)\, E_2$ is the normal at $x = \pi/\sqrt{\kappa}$. Finally we note $\nabla_{v_2} v_1 = \big((c-1)\frac{\kappa}{4\tau} + \tau\big)\, N$ at $x = \pi/\sqrt{\kappa}$, so that the shape operator is

$$S = \begin{pmatrix} 0 & (c-1)\frac{\kappa}{4\tau} + \tau \\ (c-1)\frac{\kappa}{4\tau} + \tau & 0 \end{pmatrix}.$$

For $\kappa = 4H^2 - 1$ and $\tau = H$ Daniel's correspondence and (3.8) imply

$$k_{\widetilde{v}} = H + \left\langle S \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle = 2H + (c-1)\frac{4H^2-1}{4H}.$$

We now show $k_{\widetilde{v}} > 1$:

$$\begin{aligned} k_{\widetilde{v}} &= 2H + (c-1)\frac{4H^2-1}{4H} = \frac{8H^2 + (c-1)(4H^2-1)}{4H} \\ &= \frac{8H^2 + c(4H^2-1) - (4H^2-1)}{4H} = \frac{4H^2 + 1 + c(4H^2-1)}{4H} \\ &> \frac{4H^2+1}{4H} = \frac{4H^2 - 4H + 1 + 4H}{4H} = \frac{(2H-1)^2 + 4H}{4H} > \frac{4H}{4H} = 1. \end{aligned}$$

Curves in $\mathbb{H}^2$ with constant geodesic curvature greater than 1 are rotationally invariant, which proves the claim about the sister surface of f^c.

(iii): Finally we compute which part of f^c corresponds to one half of the vertical unduloid $\widetilde{f^c}$. We need the following auxiliary result to have a reference for "one half":

Lemma 4.4. *Let $c\colon [0,L] \to \mathbb{H}^2$ be a unit-speed parametrisation of a simple closed circle in $\mathbb{H}^2$ with constant geodesic curvature $k_c > 1$. Then $R = \coth^{-1}(k_c)$ is the intrinsic radius of c.*

Proof. Let D denote the closed disc bounded by c. By Gauß-Bonnet we have

$$\int_D (-1)\, dA_{\mathbb{H}^2} + \int_c k_c\, dt = 2\pi\chi(D).$$

For the (unknown) intrinsic radius R we know $L = 2\pi\sinh(R)$ and thus the area of D in $\mathbb{H}^2$ equals $2\pi(\cosh(R) - 1)$. Since $\chi(D) = 1$ we get

$$-2\pi(\cosh(R) - 1) + 2\pi\sinh(R)k_c = 2\pi \iff k_c = \coth(R). \qquad \square$$

Proof of Theorem 4.1 continued: We know that $\widetilde{v}$ is a curve of constant geodesic curvature $k_{\tilde{v}} > 1$. Its projection onto $\mathbb{H}^2$ is therefore a curve c_R with intrinsic radius $R = \coth^{-1}(k_{\tilde{v}})$. The circumference of c_R is $2\pi\sinh(R)$, so that one half of a vertical unduloid is realised at $T = \pi\sinh(\coth^{-1}(k_{\tilde{v}}))$.

Using $\sinh(x) = \frac{\exp(x)-\exp(-x)}{2}$ and $\coth^{-1}(y) = \frac{1}{2}\ln\left(\frac{1+y}{1-y}\right)$ we obtain

$$T = \frac{\pi}{\sqrt{k_{\tilde{v}}^2 - 1}}.$$

In order to prove the claim about the horizontal fields of h_1 and h_2 we will show that

$$\mathcal{T}\colon [0,1] \to \mathbb{R}, \qquad c \mapsto (c+1)\frac{4H^2-1}{4H}\frac{\pi}{\sqrt{k_{\tilde{v}}^2 - 1}}$$

is strictly decreasing on $[0,1)$. In Example 4.3 we computed $\mathcal{T}(0) = \pi$ and $\mathcal{T}(1) \in (0,\pi)$, so that monotonicity of $\mathcal{T}$ on $[0,1)$ shows $\mathcal{T}(c) \in (0,\pi)$ for all $c \in (0,1]$.

For the monotonicity of $\mathcal{T}$ we note

$$\mathcal{T}(c) = \left(k_{\tilde{v}} - \frac{1}{2H}\right)\frac{\pi}{\sqrt{k_{\tilde{v}}^2 - 1}} = g(k_{\tilde{v}}(c)),$$

where

$$g\colon\ [k_{\tilde{v}}(0), k_{\tilde{v}}(1)) \to \mathbb{R}, \qquad g(t) := \left(t - \frac{1}{2H}\right)\frac{\pi}{\sqrt{t^2-1}}$$

is strictly decreasing and $k_{\tilde{v}}$ strictly increasing. $\square$

There are more restrictions on the boundary curves of a horizontal unduloid:

Proposition 4.5. *Let $\widetilde{\Sigma}$ be a singly periodic properly (Alexandrov) embedded* MC*H-annulus in $\mathbb{H}^2 \times \mathbb{R}$ with $H > \frac{1}{2}$ and horizontal axis $\tilde{\gamma}$. Then the sister curves h_1 and h_2 from the intersection of $\widetilde{\Sigma}$ with its vertical mirror plane are horizontal geodesics and their respective horizontal fields F_1 and F_2 satisfy $F_1 = -F_2$.*

Proof. The surface $\widetilde{\Sigma}$ has a horizontal mirror plane P by Proposition 3.14 (ii). By Proposition 3.10 the reflection through P corresponds to a geodesic reflection about a vertical geodesic v contained in the minimal sister surface $\Sigma \subset \mathbb{S}^3(4H^2-1, H)$. The surface Σ is invariant under this rotation, which implies $F_1 = -F_2$. $\square$

Example 4.6 (Horizontal unduloid)**.** Let us consider the converse of Proposition 4.5: Given horizontal geodesics h_1 and h_2 in $\mathbb{S}^3(4H^2-1, H)$ with horizontal fields F_1 and F_2 satisfying $F_1 = -F_2$. Do they bound a minimal surface whose sister surface is one half of a horizontal unduloid? They do if h_1 and h_2 are not too far away from each other.

The idea is to use the construction of Manzano and Torralbo in [**MT14**]. They constructed one quarter of a horizontal unduloid by solving a Plateau problem in the Berger sphere $\mathbb{S}^3(4H^2-1, H)$. The boundary curve consists of segments of three horizontal geodesics and one vertical geodesic; the Hopf projection is a convex sector in $\mathbb{S}^2(4H^2-1)$, so that there is a unique graphical solution of the Plateau problem.

For $\kappa > 0$ and $\tau \in \mathbb{R}$ let $\lambda \in \left[0, \frac{\pi}{2\sqrt{\kappa}}\right]$. Moreover let h_1 and h_2 be the curves

$$h_1(t) = \begin{pmatrix} \cos\left(\frac{\sqrt{\kappa}}{2}t\right) \\ \sin\left(\frac{\sqrt{\kappa}}{2}t\right) \end{pmatrix},$$

$$h_2(t) = \begin{pmatrix} \cos(\sqrt{\kappa}\lambda)\cos\left(\frac{\sqrt{\kappa}}{2}t\right) + i\sin(\sqrt{\kappa}\lambda)\sin\left(\frac{\sqrt{\kappa}}{2}t\right) \\ -\cos(\sqrt{\kappa}\lambda)\sin\left(\frac{\sqrt{\kappa}}{2}t\right) + i\sin(\sqrt{\kappa}\lambda)\cos\left(\frac{\sqrt{\kappa}}{2}t\right) \end{pmatrix}.$$

In view of (3.6) we notice the following: h_1 is a horizontal geodesic through $(1,0)$ with horizontal field $F_1 = E_1$ and h_2 is a horizontal geodesic through $(\cos(\sqrt{\kappa}\lambda), i\sin(\sqrt{\kappa}\lambda))$ with horizontal field $F_2 = -E_1$. We claim that h_1 and h_2 have the following properties:

(a) There is a vertical geodesic v joining $h_1\left(\frac{\pi}{2\sqrt{\kappa}}\right)$ and $h_2\left(-\frac{\pi}{2\sqrt{\kappa}}\right)$. The length ℓ of v is in the interval $\left[0, \frac{2\tau}{\kappa}\pi\right]$. Thus h_1 and h_2 are, up to a left-translation, a reparametrisation contained in a spherical helicoid f^c for $c \in [0,1]$; compare with Theorem 4.1.

(b) There is a closed geodesic polygon $\Gamma_\lambda = \gamma_1 \oplus \gamma_2 \oplus \gamma_3 \oplus \gamma_4$ such that $\gamma_1 = h_1|_{[0,\pi/\sqrt{\kappa}]}$, γ_2 and γ_4 are horizontal geodesics intersecting γ_1 orthogonally, and γ_3 is a vertical geodesic joining γ_2 and γ_4. The Hopf projection $\Pi \circ \Gamma_\lambda$ is a convex sector in $\mathbb{S}^2(\kappa)$ and by [**MT14**] it bounds a unique minimal graph Σ_λ (graph with respect to Π).

(c) Geodesic reflection across γ_3 maps h_1 onto h_2. Reflections across γ_2 and γ_4 extend the surface Σ_λ to a minimal annulus bounded by h_1 and h_2.

For (a) we exhibit v explicitly. We note first

$$h_1\left(\frac{\pi}{2\sqrt{\kappa}}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \quad \text{and} \quad h_2\left(-\frac{\pi}{2\sqrt{\kappa}}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix}\exp\left(-i\sqrt{\kappa}\lambda\right)\\ \exp\left(i\sqrt{\kappa}\lambda\right)\end{pmatrix}.$$

The curve

$$v(s) = \frac{1}{\sqrt{2}}\begin{pmatrix}\exp\left(-i\frac{\kappa}{4\tau}s\right)\\ \exp\left(i\frac{\kappa}{4\tau}s\right)\end{pmatrix}$$

is a vertical geodesic, see also (3.5), that joins $v(0) = \frac{1}{\sqrt{2}}(1,1)$ and

$$v(\ell) = h_2\left(-\frac{\pi}{2\sqrt{\kappa}}\right) \text{ for } \ell = \frac{4\tau}{\sqrt{\kappa}}\lambda.$$

By definition of λ we have $0 \leq \ell \leq \frac{2\tau}{\kappa}\pi$, as claimed.

For (b) we define curves γ_1 to γ_4 as follows:

$$\begin{aligned}
\gamma_1(t) &= \begin{pmatrix}\cos\left(\frac{\sqrt{\kappa}}{2}t\right)\\ \sin\left(\frac{\sqrt{\kappa}}{2}t\right)\end{pmatrix}, && t \in \left[0, \frac{\pi}{\sqrt{\kappa}}\right],\\
\gamma_2(t) &= \begin{pmatrix}\cos\left(\frac{\sqrt{\kappa}}{2}t\right)\\ i\sin\left(\frac{\sqrt{\kappa}}{2}t\right)\end{pmatrix}, && t \in [0, \lambda],\\
\gamma_3(t) &= \begin{pmatrix}\exp\left(i\frac{\kappa}{4\tau}t\right)\cos\left(\frac{\sqrt{\kappa}}{2}\lambda\right)\\ \exp\left(-i\frac{\kappa}{4\tau}t\right) i\sin\left(\frac{\sqrt{\kappa}}{2}\lambda\right)\end{pmatrix}, && t \in \left[0, \frac{2\tau}{\kappa}\pi\right],\\
\gamma_4(t) &= \begin{pmatrix}i\sin\left(\frac{\sqrt{\kappa}}{2}t\right)\\ \cos\left(\frac{\sqrt{\kappa}}{2}t\right)\end{pmatrix}, && t \in \left[0, \frac{\pi}{\sqrt{\kappa}} - \lambda\right].
\end{aligned}$$

One can check that this defines a closed polygon Γ_λ with orientations as in Figure 4.2 on page 60. Looking at (3.6) and (3.5) we see the following: γ_1 is E_1-circle, γ_2 and γ_4 are E_2-circles (in particular they intersect γ_1 orthogonally) and γ_3 is a vertical geodesic. By choice of λ the sector $\Pi \circ \Gamma_\lambda$ is convex.

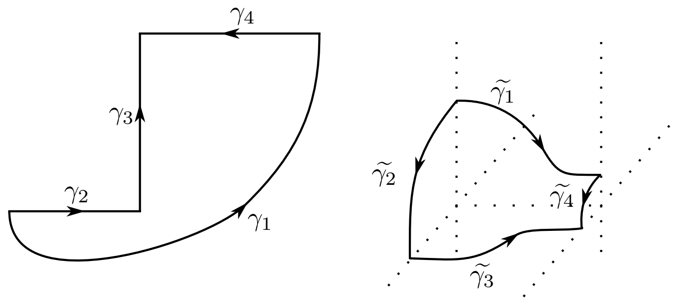


FIGURE 4.2. Qualitative sketches of Γ_λ on the left and of the sister contour $\widetilde{\Gamma_\lambda}$ on the right side; compare also with [**MT14**, Figure 3]

Finally we check (c). Let $\rho_0(z,w) := (z,-w)$ denote the geodesic reflection across the vertical geodesic

$$v_0(t) := \begin{pmatrix} \exp\left(i\frac{\kappa}{4\tau}t\right) \\ 0 \end{pmatrix}$$

through $(1,0)$. A computation shows $\gamma_3(t) = \mathcal{L}_{\gamma_2(\lambda)}(v_0(t))$, so that

$$\rho(z,w) := \left(\mathcal{L}_{\gamma_2(\lambda)} \circ \rho_0 \circ \mathcal{L}_{(\gamma_2(\lambda))^{-1}}\right)(z,w)$$

is the geodesic reflection across γ_3. Another computation yields $h_2(t) = \rho(h_1(t))$. Reflections across γ_2 and γ_4 extend the surface to a minimal annulus bounded by h_1 and h_2: This is because the length of the segment γ_1 is one quarter of the length of h_1 or h_2, and thus we go once around h_1 and h_2, respectively.

To finish this section, we make the following observation:

Remark 4.7. Let h_1 and h_2 be as in Example 4.6. For $H > 1/2$ let $\kappa = 4H^2 - 1$ and $\tau = H$. We have seen that there is a minimal surface bounded by h_1 and h_2 such that its sister surface in $\mathbb{H}^2 \times \mathbb{R}$ is one half of a horizontal unduloid, that is, it extends to a horizontal unduloid. On the other hand h_1 and h_2 bound a spherical helicoid f^c that corresponds to a part of vertical unduloid in $\mathbb{H}^2 \times \mathbb{R}$ (not necessarily one half of it). Thus the curves h_1 and h_2 bound at least two minimal annuli, which have a different geometric meaning for the MCH-sister surface. Combining all the results up to this point we see that a *tilted unduloid* in $\mathbb{H}^2 \times \mathbb{R}$ can only exist if we have a multiple solution theorem for minimal annuli bounded by any pair of linked horizontal geodesics in $\mathbb{S}^3(4H^2-1, H)$.

4.2. Unduloids in $\mathbb{R}^3$ and their sisters in $\mathbb{S}^3$

To motivate the following section, let us be more precise on the conjectured tilted unduloids in $\mathbb{H}^2 \times \mathbb{R}$. For $\alpha \in [0, \pi/2]$ consider a geodesic $\tilde{\gamma}_\alpha$ in $\mathbb{H}^2 \times \mathbb{R}$ with slope $\cos(\alpha) = \langle \tilde{\gamma}', \xi \rangle$. We believe that a tilted unduloid with axis $\tilde{\gamma}_\alpha$ arises as a one-parameter family

$$\left(\widetilde{\Sigma}^{c,\alpha}\right)_{c \in [0,1]}$$

where c is supposed to correspond to the neck size; $c = 0$ represents a chain of MCH-spheres and $c = 1$ a tilted MCH-cylinder. All α are supposed to result in different one-parameter families since $\mathbb{H}^2 \times \mathbb{R}$ is not isotropic. We therefore expect a *two-parameter family* of surfaces

$$\left(\widetilde{\Sigma}^{c,\alpha}\right)_{(c,\alpha) \in [0,1] \times [0,\pi/2]} .$$

In $\mathbb{R}^3$ all axes are equivalent, but nevertheless we can regard $\mathbb{R}^3$ as a Riemannian fibration and thus distinguish vertical and horizontal vectors. Any unduloid in $\mathbb{R}^3$ has a vertical mirror plane containing the axis of rotation and the mirror curves in this plane correspond to horizontal geodesics h_1 and h_2 in $\mathbb{S}^3$ in the terminology of Chapter 3. Let us fix this plane and rotate about the normal of the plane so that one point of the axis is fixed. The mirror plane is invariant under this rotation, but the axis of the unduloid tilts within the mirror plane. We thus have a *two-parameter family* of unduloids: The parameters are the neck size n and the angle α the axis encloses with the vertical direction.

In this section we exhibit the corresponding two-parameter family in $\mathbb{S}^3$. We compute an explicit parametrisation of the minimal sister surface in order to describe the action in $\mathbb{S}^3$ corresponding to tilting an unduloid in $\mathbb{R}^3$.

4.2.1. Vertical unduloids and spherical helicoids. In order to compute a parametrisation of the sister surface of a vertical unduloid in $\mathbb{R}^3$ we will use the relation of sister curves in the Lawson correspondence, that is, we will use Proposition 3.11. Recall that we choose the coordinate system in $\mathbb{R}^3$ as in Figure 3.1 on page 49 and think of $\boldsymbol{i}$ as a vertical direction. A vertical unduloid is an MC1-surface of revolution about the $\boldsymbol{i}$-axis; it has neck size n in $(0, \pi]$, which is the circumference of the shortest non trivial closed geodesic. We denote a parametrisation of it by $\tilde{f}(x, y)$ and consider the following situation:

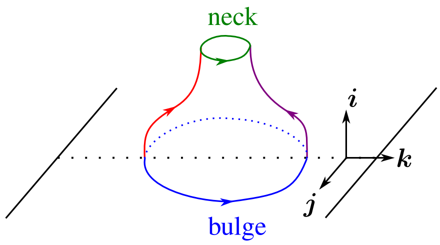


FIGURE 4.3. Parametrisation of vertical unduloid in $\mathbb{R}^3$; bulge and neck are contained in planes with normal $\boldsymbol{i}$

The "bulge" corresponds to an $\boldsymbol{i}$-Hopf circle in $\mathbb{S}^3$. Without loss of generality it is parametrised by

$$f(0,y) = \cos((2\pi - n)y)\mathbf{1} + \sin((2\pi - n)y)\boldsymbol{i}.$$

The meridians of a vertical unduloid lie in vertical planes perpendicular to the conormal along the bulge. This conormal rotates and is described by

$$u(y) = \cos(2\pi y)\boldsymbol{j} + \sin(2\pi y)\boldsymbol{k}, \qquad y \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right].$$

Therefore the meridians correspond to $u(y)$-Hopf circles through $f(0,y)$ in $\mathbb{S}^3$ which can be parametrised by

$$x \mapsto f(0,y)(\cos(x) + \sin(x)u(y)).$$

Then the cousin in $\mathbb{S}^3$ is parametrised by

$$f^n(x,y) = \begin{pmatrix} \cos(x)\exp(-iny) \\ \sin(x)\exp(i(2\pi - n)y) \end{pmatrix}.$$

We see that f^n agrees with the spherical helicoid from Proposition 3.5 for $\kappa = 4$, $\tau = 1$ with parameters $\varphi = \pi$ and $\ell = \frac{n}{2}$. For $n = \pi$ we get cylinders in $\mathbb{R}^3$ and for $n = 0$ a chain of spheres.

4.2.2. Tilting an unduloid. Suppose that in $\mathbb{R}^3$ we rotate the $\boldsymbol{i}$-axis in the $\boldsymbol{i}$-$\boldsymbol{k}$-plane to obtain a tilted unduloid and consider the symmetry curves in the vertical mirror plane, perpendicular to $\boldsymbol{j}$. Then the horizontal geodesics corresponding to them are $\boldsymbol{j}$-Hopf circles. In $\mathbb{S}^3$ the $\boldsymbol{j}$-Hopf field E_1 is a Killing vector field so that the flow of E_1 acts by a one-parameter family of isometries. We claim that this flow in $\mathbb{S}^3$ corresponds to tilting an unduloid in $\mathbb{R}^3$.

We define the family $(\Phi_\alpha)_{\alpha\in\mathbb{R}}$ by

$$\Phi_\alpha\colon \mathbb{S}^3 \to \mathbb{S}^3, \qquad \Phi_\alpha(z,w) := \begin{pmatrix} \cos\left(\frac{\alpha}{2}\right) z - \sin\left(\frac{\alpha}{2}\right) w \\ \sin\left(\frac{\alpha}{2}\right) z + \cos\left(\frac{\alpha}{2}\right) w \end{pmatrix}.$$

This family satisfies $\frac{d}{d\alpha}\Phi_\alpha(z,w) = \frac{1}{2}E_1$, i.e., $(\Phi_\alpha)_{\alpha\in\mathbb{R}}$ is the flow of a Killing vector field on $\mathbb{S}^3$ and thus a one-parameter family of isometries.

Theorem 4.8. *Let $f_\alpha^n := \Phi_\alpha \circ f^n \colon \mathbb{R}\times[0,1/2] \to \mathbb{S}^3$. Then f_α^n is a minimal surface and the sister surface in $\mathbb{R}^3$ under the Lawson correspondence as in Proposition 3.8 is the half of an unduloid (to one side of a vertical mirror plane perpendicular to $\boldsymbol{j}$). This unduloid has a tilted axis. The curves $h_1 := f_\alpha^n(\cdot,0)$ and $h_2 := f_\alpha^n(\cdot,1/2)$ are $(\pm\boldsymbol{j})$-Hopf circles respectively horizontal geodesics with horizontal field $\pm E_1$. Their Hopf projections $\Pi\circ h_1$ and $\Pi\circ h_2$ enclose an angle of $\pi - n$.*

In the proof we only assume knowledge regarding a vertical unduloid in $\mathbb{R}^3$ and its sister surface in $\mathbb{S}^3$ in order to use arguments that could apply to the existence problem for tilted unduloids in $\mathbb{H}^2\times\mathbb{R}$.

Proof. It is well-known that the sister surface of f_0^n is one half of vertical unduloid, see [**GB93**]. In fact, this can be proved exactly like Theorem 4.1 (ii). The necks and bulges of it correspond to vertical geodesics. The curve

$$v\colon [0,1/2] \to \mathbb{S}^3, \qquad v(s) := f_0^n(0,s) = \begin{pmatrix} \exp(-ins) \\ 0 \end{pmatrix}$$

traces out the axis of the helicoid and joins h_1 and h_2; it has length $\ell = L(v) = \frac{n}{2}$. Thus $\Pi\circ h_1$ and $\Pi\circ h_2$ enclose an angle of $\pi - n$ by Proposition 3.5. Moreover, the claim about the horizontal field of h_1 and h_2 follows from a look at the construction of f^n.

Now we consider

$$c_\alpha(s) := \Phi_\alpha(v(s)) = \begin{pmatrix} \cos\left(\frac{\alpha}{2}\right)\exp(-ins) \\ \sin\left(\frac{\alpha}{2}\right)\exp(-ins) \end{pmatrix}.$$

In view of (3.3) and using addition theorems for sine and cosine we obtain

$$c_\alpha'(s) = \begin{pmatrix} -in\cos\left(\frac{\alpha}{2}\right)\exp(-ins) \\ -in\sin\left(\frac{\alpha}{2}\right)\exp(-ins) \end{pmatrix} = -n\sin(\pi-\alpha)E_2 + n\cos(\pi-\alpha)\xi.$$

This shows that c_α is a geodesic with slope $\pi - \alpha$ in $\mathbb{S}^3$, that is, c_α is u-Hopf circle with $u = -\sin(\pi - \alpha)\boldsymbol{k} + \cos(\pi - \alpha)\boldsymbol{i}$. Thus the sister curve of c_α lies in a plane with normal $(0, -\sin(\pi - \alpha), \cos(\pi - \alpha))$. We know that h_1 and h_2 are invariant under the flow $(\Phi_\alpha)_{\alpha\in\mathbb{R}}$, so that the sister curves $\widetilde{h_1}$ and $\widetilde{h_2}$ are contained in vertical mirror planes with normal $\boldsymbol{j}$. We show that these planes are equal. Let η_{c_α} and η_v denote the conormals along c_α and v, respectively. Then we have

$$\begin{aligned}\int_0^{1/2} \left\langle \widetilde{c_\alpha}', \boldsymbol{j} \right\rangle_{\mathbb{R}^3} ds &= \int_0^{1/2} g_{4,1}(\eta_{c_\alpha}, E_1|_{c_\alpha})\, ds \\ &= \int_0^{1/2} g_{4,1}(d\Phi_\alpha \eta_v, d\Phi_\alpha E_1|_v)\, ds \\ &= \int_0^{1/2} g_{4,1}(\eta_v, E_1)\, ds = 0.\end{aligned}$$

Thus $\widetilde{h_1}$ and $\widetilde{h_2}$ lie in the same vertical plane of $\mathbb{R}^3$, so that we can extend the surface to an unduloid with axis $(0, -\sin(\pi - \alpha), \cos(\pi - \alpha))$. $\square$

Observation. *This theorem shows that the family of tilted unduloids $\widetilde{\Sigma}_{n,\alpha}$ in $\mathbb{R}^3$ with respect to a fixed mirror plane corresponds to the family $f^n_\alpha = \Phi_\alpha \circ f^n$ of spherical helicoids with tilted axes in $\mathbb{S}^3$. The arguments rely on $(\Phi_\alpha)_{\alpha\in\mathbb{R}}$ being a one-parameter family of isometries of $\mathbb{S}^3$.*

Changing the metric of $\mathbb{S}^3$ to that of $\mathbb{S}^3(\kappa, \tau)$ with $\kappa \neq 4\tau^2$ we see that Φ_α is a diffeomorphism of $\mathbb{S}^3(\kappa, \tau)$ but not an isometry; this is because E_1 is not a Killing vector field in $\mathbb{S}^3(\kappa, \tau)$. Nevertheless, $\Phi_{\pi/2}$ maps a vertical geodesic onto a horizontal geodesic (changing the length, of course). Therefore $\Phi_{\pi/2}$ yields the polygon Γ_λ used in Example 4.6 to construct one quarter of a horizontal unduloid in $\mathbb{H}^2 \times \mathbb{R}$.

CHAPTER 5

A minimax principle in the Berger spheres

Finding minimal surfaces bounded by two closed curves is an interesting problem in itself. The solution depends on the topology of the surface we are looking for. We are interested in minimal annuli. For our particular problem, minimal annuli bounded by linked horizontal geodesics in $\mathbb{S}^3(\kappa, \tau)$, two questions arise: Is there a minimal annulus bounded by these two curves? How many solutions exist?

We have already answered the first question: in the Berger spheres we have exhibited one explicit solution in Proposition 3.5.

Regarding the second question, we have seen in Example 4.6 that certain horizontal geodesics h_1 and h_2 with horizontal fields satisfying $F_1 = -F_2$ bound at least two minimal annuli. For a general pair of linked horizontal geodesics h_1 and h_2 in the Berger spheres the question regarding the number of minimal annuli bounded by h_1 and h_2 is more delicate. In a series of papers, see [**Min89, Min93b, Min93a**], Ji Min proves a minmax principle to obtain a multiple solution theorem for a pair of linked curves in $\mathbb{S}^3$, or more generally in any compact Riemannian 3-manifold.

We start this chapter by recalling an argument used in [**TY90**] to show that a pair of closed and linked curves in a Riemannian 3-manifold satisfies the *Douglas condition*, that is, these curves bound an annulus that has less area than the sum of area-minimising discs. In Section 5.2 we introduce all necessary notations and definitions to formulate the minimax principle by Ji Min. In Appendix B we provide the details of these papers because the techniques specified therein are interesting themselves and the paper has been unknown to fellow mathematicians. In Section 5.3 we show that this multiple solution theorem can be applied to our setup, i.e., we get at least two minimal annuli for a given pair of linked horizontal geodesics in the Berger spheres. We thus obtain many new minimal annuli in the Berger spheres.

5.1. Minimal annuli bounded by closed and linked curves

Horizontal geodesics in the Berger spheres are linked if they do not intersect. A general result for Riemannian 3-manifolds allows us to verify the *Douglas condition* for linked and closed curves. We first define this condition:

Definition. Let M be a simply connected complete Riemannian 3-manifold of bounded geometry, that is, there is an upper bound for the absolute value of the sectional curvature and a positive lower bound for the injectivity radius of M (obviously satisfied for all $E(\kappa, \tau)$-spaces). Let Γ_1 and Γ_2 be two closed, rectifiable, disjoint Jordan curves in M. We say that Γ_1 and Γ_2 satisfy the *Douglas condition* if there exists an annulus bounded by Γ_1 and Γ_2 that has less area than the sum of area-minimising minimal discs bounded by Γ_1 and Γ_2, respectively.

The minimal discs, that is, solutions to the Plateau problem for Γ_1 and Γ_2, respectively, exist since M is simply connected and has bounded geometry; see [**Jos91**]. For $M = \mathbb{R}^3$ the Douglas condition is sufficient to find a minimal annulus bounded by Γ_1 and Γ_2. In a general Riemannian 3-manifold a sequence of annuli satisfying the Douglas condition and minimising the Dirichlet energy could split into a minimal annulus and a minimal sphere. In order to prevent this from happening a generalised Douglas condition is needed. For such a generalisation we refer to [**Jos91**, Theorem 4.7.1].

In the proof of the minimax principle we need the following property:

Proposition 5.1 (Douglas condition)**.** *Suppose M is a simply connected complete Riemannian* 3*-manifold of bounded geometry and let* Γ_1 *and* Γ_2 *be two closed, rectifiable, disjoint Jordan curves in M which are linked. Then* Γ_1 *and* Γ_2 *satisfy the Douglas condition in M, for any choice of orientations of* Γ_1 *and* Γ_2.

Proof. We follow [**TY90**, Lemma 2.1]. Let Σ_1 and Σ_2 be area-minimising solutions to the Plateau problem for Γ_1 and Γ_2, respectively. These solutions exist for the reason mentioned above. We note that Σ_1 and Σ_2 are immersed in the interior. Since Γ_1 and Γ_2 are linked there is $p \in \Sigma_1 \cap \Sigma_2$ such that Σ_1 and Σ_2 intersect transversally. For $j \in \{1, 2\}$ we consider embedded pieces $\Sigma'_j \subset \Sigma_j$ along which the intersection is transverse at p. We can choose these pieces sufficiently small so that there is a neighbourhood $N = N(p)$ of p with $\Sigma'_1, \Sigma'_2 \subset N$ and a chart $x \colon N \to U$ where $U \subset \mathbb{R}^3$ is open. We endow U with the metric induced by M, that is, x is an isometry.

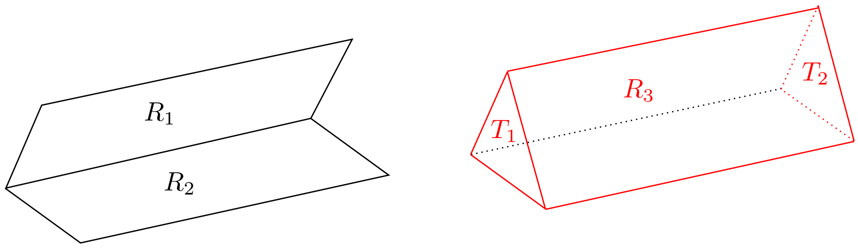


FIGURE 5.1. Local picture around $\Phi(p)$

If these pieces are sufficiently small we can find a diffeomorphism F from a neighbourhood of $u := x(p)$ onto itself such that $F(u) = u$, F is an isometry at u and F maps $x(\Sigma_j')$ into a plane P_j. Now consider two congruent rectangles $R_j \subset F(x(\Sigma_j'))$ which have a common side L in $P_1 \cap P_2$. Then R_1, R_2, a third rectangle R_3 and two triangles T_1 and T_2 together form the boundary of a prism as shown above in Figure 5.1.

We will replace $R_1 \cup R_2$ by $R := T_1 \cup R_3 \cup T_2$. This can be done in such a way that R has less area than $R_1 \cup R_2$ (the length of L has to be sufficiently small). Using this replacement, composed with $(F \circ x)^{-1}$, and the fact that $\Phi := F \circ x$ is an isometry at p we see the following: $\Sigma = \big(\Sigma_1 \setminus \Phi^{-1}(R_1)\big) \cup \big(\Sigma_2 \setminus \Phi^{-1}(R_2)\big) \cup \Phi^{-1}(R)$ has less area than the sum of areas of Σ_1 and Σ_2, i.e., the Douglas condition is satisfied. In order to see that Σ is an annulus it is useful to have a look at Figure 5.2 on page 68.

Since Γ_1 and Γ_2 are linked, this construction can be carried out so that the orientations of the boundary curves are respected. To be more precise, there are four rectangles intersecting in L and lying in P_1 or P_2, respectively. The rectangles have to be chosen accordingly to the orientations of Γ_1 and Γ_2. It order to see this it is useful to draw the possible orientations in Figure 5.2. $\square$

5.2. Minimax principle for minimal annuli

In [**Min93a**] Ji Min obtained a multiple solution theorem for the Douglas problem of finding a minimal annulus bounded by two curves in a compact Riemannian 3-manifold. A perturbation method as in [**Uhl81**] is used and applied to an index generalising the Lusternik-Schnirelman category. This index is then a lower bound for the number of minimal annuli spanned by these two curves. In Subsection 5.2.1 and Subsection 5.2.2 we provide summaries of [**Min93a**].

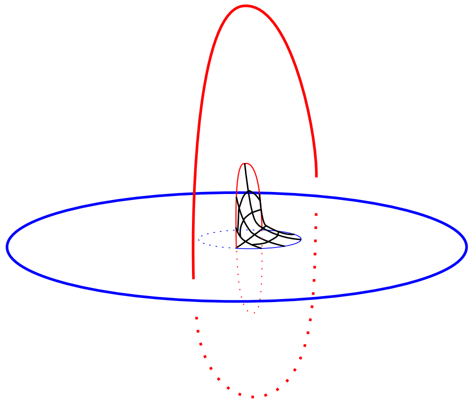


FIGURE 5.2. Sketch of the replacement for linked planar curves Γ_1 and Γ_2 in $\mathbb{R}^3$

Let M denote a simply connected compact 3-manifold and let Γ_1 and Γ_2 be closed, rectifiable Jordan curves in M. We endow M with a Riemannian metric g such that (M, g) is isometrically embedded in $\mathbb{R}^k$ for some $k \in \mathbb{N}$, the curves Γ_1 and Γ_2 are geodesics in (M, g), which is possible since M is compact, and they are a positive distance apart. In this setting, $\exp_p\colon T_pM \to M$ maps $T_p\Gamma_j$ to Γ_j when $p \in \Gamma_j$ for $j \in \{1, 2\}$.

Furthermore we will use the following notations:

$$\begin{aligned} D &:= \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 < 1\}, \\ A_\rho &:= A_{\rho,1} := \{(x, y) \in \mathbb{R}^2 \colon \rho < x^2 + y^2 < 1\}, \\ C_\rho &:= \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 = \rho\}. \end{aligned}$$

A minimal surface of annulus type bounded by Γ_1 and Γ_2 is characterised by a map $u \in C^2(A_\rho, M) \cap C^0(\overline{A_\rho}, M)$ for some $\rho \in (0, 1)$ such that the following is satisfied:

(MA1) $\Delta u = A(u)(du, du)$ in A_ρ,

(MA2) $|u_x|^2 - |u_y|^2 = 0 = u_x \cdot u_y$ in A_ρ,

(MA3) $u|_{C_\rho}$ and $u|_{\partial D}$ are weakly monotone parametrisations of Γ_1 and Γ_2, respectively.

Here $A(u)$ denotes the second fundamental form of M in $\mathbb{R}^k$. The first property describes the harmonicity of u, the second one the conformity and the third one is a boundary condition. We note that u may have branch points.

5.2.1. Variational approach. For $p > 2$ the Sobolev space $W^{1,p}(A_\rho, \mathbb{R}^k)$ is embedded in $C^0(A_\rho, \mathbb{R}^k)$ so that the following definition makes sense:

Definition. For $p > 2$ we consider the spaces

$$W^{1,p}_{\Gamma_1,\Gamma_2}(A_\rho, M) := \big\{u \in W^{1,p}(A_\rho, \mathbb{R}^k) \colon u(A_\rho) \subset M, u(C_\rho) \subset \Gamma_1, u(\partial D) \subset \Gamma_2 \text{ and } \deg(u|_{C_\rho}) = 1 = \deg(u|_{\partial D})\big\}, \tag{5.1}$$

$$X(\rho) := X_p(\rho) := \Big\{u \in W^{1,p}_{\Gamma_1,\Gamma_2}(A_\rho, M) \colon u \text{ satisfies (MA3)}\Big\}. \tag{5.2}$$

We consider $(u, \rho) \in W^{1,p}_{\Gamma_1,\Gamma_2}(A_{\rho_0}, M) \times (0, 1)$ and set

$$u^\rho(r, \theta) := u\left(\frac{\rho_0 - \rho + (1 - \rho_0)r}{1 - \rho}, \theta\right). \tag{5.3}$$

Then we have $u^\rho \in W^{1,p}_{\Gamma_1,\Gamma_2}(A_\rho, M)$. Furthermore we define the functional

$$F(u, \rho) := E(u^\rho) = \frac{1}{2}\int_{A_\rho} |\nabla u^\rho|^2 \, dx \, dy.$$

Critical points of this functional satisfy (MA1) to (MA3), that is, they are minimal annuli bounded by Γ_1 and Γ_2. The main problem is that F does not satisfy the Palais-Smale condition and thus one has to find another way to show that a minimising sequence converges to a critical point. The parameter $\rho \in (0, 1)$ is the conformal parameter.

Furthermore we define the spaces $X = X_p$ also for $p = 2$. Since we do not have an embedding into a space of continuous functions for $p = 2$, we must modify the definition as follows compared to the case $p > 2$:

Definition. Let

$$\begin{aligned}
Y_j &:= \{\varphi \in C^0(\mathbb{S}^1, \Gamma_j) \colon \varphi \text{ is weakly monotone}\}, \qquad j \in \{1, 2\},\\
W^{1,2}_\varphi &:= \{u \in W^{1,2}(D, \mathbb{R}^k) \colon u(D) \subset M \text{ and } u|_{\partial D} = \varphi \text{ almost everywhere}\},\\
W^{1,2}_{\varphi,\psi} &:= \{u \in W^{1,2}(A_{\rho_0}, \mathbb{R}^k) \colon u(A_{\rho_0}) \subset M, u(\rho_0, \cdot) = \varphi, u(1, \cdot) = \psi \text{ a.e.}\}.
\end{aligned}$$

Define

$$\begin{aligned}
X_2(\Gamma_1, \Gamma_2) &:= \bigcup_{\varphi \in Y_1} \bigcup_{\psi \in Y_2} W^{1,2}_{\varphi,\psi},\\
X_2(\Gamma_j) &:= \bigcup_{\varphi \in Y_j} W^{1,2}_\varphi, \qquad j \in \{1, 2\}.
\end{aligned}$$

We also set

$$\begin{aligned} m_j &= m_j(\Gamma_j) = \inf\{E(u)\colon u \in X_2(\Gamma_j)\}, \qquad j \in \{1,2\},\\ m^* &= m^*(\Gamma_1,\Gamma_2) = \inf\{F(u,\rho)\colon (u,\rho) \in X_2(\Gamma_1,\Gamma_2)\times(0,1)\},\\ d^* &= d^*(\Gamma_1,\Gamma_2) = m_1 + m_2. \end{aligned}$$

Finally we define

$$s_0 = s_0(M) = \begin{cases} \inf\{E(u) \mid u\colon \mathbb{S}^2 \to M \text{ is harmonic and non-constant}\},\\ \infty \text{ if } M \text{ admits no minimal sphere.} \end{cases}$$

5.2.2. Index and minimax principle. The last ingredient we need is a topological index generalising the Lusternik-Schnirelman category.

Let $Z := W^{1,p}_{\Gamma_1,\Gamma_2}$ and for $u \in Z$ define

$$[u] := \{u \circ e^{i\theta}\colon \theta \in (0,2\pi]\}.$$

We say that $B \subset Z \times (0,1)$ is *deformable* to $[u] \times \{\rho\}$ in Z for some $(u,\rho) \in Z \times (0,1)$ if there exists a continuous map $\mathcal{H}\colon [0,1] \times B \to Z \times (0,1)$ such that

$$\mathcal{H}(0,\cdot) = \mathrm{id} \quad \text{and} \quad \mathcal{H}(1,B) \subset [u] \times \{\rho\}.$$

Definition. For $A \subset Z \times (0,1)$ we set

$$\begin{aligned} i(A) := \inf\{k \geq 0\colon & \text{ there exist } k \text{ closed subsets } B_1,\dots,B_k \subset Z \times (0,1)\\ & \text{such that } A \subset B_1 \cup \cdots \cup B_k \text{ and } B_j \text{ is deformable to } [u_j] \times \{\rho_j\} \text{ in } Z \times (0,1)\\ & \text{where } (u_j,\rho_j) \in Z \times (0,1) \text{ for } j \in \{1,\dots,k\}\}. \end{aligned}$$

We call $i(A)$ *index* of A in $Z \times (0,1)$. To emphasise this dependence we sometimes denote $i(A)$ by $i(A, Z \times (0,1))$.

Since $X \times (0,1) \subset Z \times (0,1)$ we set

$$c_\ell := \inf\left\{\sup_A F\colon A \subset X \times (0,1) \text{ and } i(A, Z \times (0,1)) \geq \ell\right\}$$

where $\ell \in [1, i(X \times (0,1), Z \times (0,1))]$.

The following theorem tells us under which conditions c_ℓ is a critical value of F and how many critical points with this critical value exist. It is a Lusternik-Schnirelman type of theorem.

Theorem 5.2 ([**Min93a**, Theorem 4.6]). *The following statements are true:*

(i) If $c_\ell < \min(m^* + s_0, d^*)$ then c_ℓ is a critical value of F.

(ii) If $c_\ell = c_{\ell+1} = \cdots = c_{\ell+m} = c < \min(m^* + s_0, d^*)$ and c is not a limit of critical values of F then $i(K_c) \geq m+1$ where K_c denotes the set of critical points of F with value c.

If c_1 to c_ℓ are all distinct then we have ℓ minimal annuli. If some of these values are equal and that value is a limit point of critical values then F has infinitely many critical points. Finally, the last case is covered by (ii) of the theorem. Therefore we get the following corollary:

Corollary 5.3 ([**Min93a**, Corollary 4.7]). *If*

$$c_\ell < \min(m^* + s_0, d^*) \textit{ for some } \ell \in [1, i(X \times (0,1))],$$

then there exist ℓ minimal annuli with areas $c_1, c_2, \ldots, c_\ell$ bounded by Γ_1 and Γ_2 in M.

Remark 5.4. (a) The condition $c_\ell < \min(m^* + s_0, d^*)$ generalises the Douglas condition. If $s_0 = \infty$ then it is precisely the Douglas condition. The condition $c_\ell < m^* + s_0$ guarantees that a minimising sequence does not split into an annulus and a sphere.

(b) In [**Min93a**, Corollary 4.7] there is also a claim about the existence of infinitely many minimal annuli bounded by Γ_1 and Γ_2: If two values from c_1 to c_ℓ are equal then there are infinitely many minimal annuli bounded by Γ_1 and Γ_2. This claim is without proof and thus we did not include it.

5.3. Multiple solution theorem in Berger spheres

We show that any pair of linked horizontal geodesics h_1 and h_2 in $\mathbb{S}^3(\kappa, \tau)$ bound at least two minimal annuli bounded by these curves:

Theorem 5.5. *Let h_1 and h_2 be horizontal geodesics $\mathbb{S}^3(\kappa, \tau)$. Then there are at least two (possibly branched) minimal annuli bounded by h_1 and h_2 for a prescribed orientation of the boundary curves.*

Proof. We apply Corollary 5.3 to this situation. The assumptions on the ambient manifold and the boundary curves are satisfied: $\mathbb{S}^3(\kappa, \tau)$ is compact and h_1 and h_2 are geodesics. It remains to show $c_2 < \min(m^* + s_0, d^*)$. We proceed as follows:

(i) We show $\min(m^* + s_0, d^*) = d^*$, that is, we only have to check the Douglas condition.
(ii) We construct a subset $A \subset X \times (0,1)$ such that

$$i(A) \geq 2 \quad \text{and} \quad \sup_{(u,\rho)\in A} F(u,\rho) < d^*.$$

To prove (i), let h be a horizontal geodesic in $\mathbb{S}^3(\kappa,\tau)$ and let Σ be a solution of the Plateau problem with respect to h, that is, Σ is a minimal disc bounded by h. It is immersed and has no branch points (neither in interior nor on the boundary since h admits a rotation of angle π). By Schwarz reflection we can extend Σ to an immersed minimal sphere S in $\mathbb{S}^3(\kappa,\tau)$. According to [**MMPR13**, Theorem 7.1], immersed minimal spheres are embedded and unique up to an isometry of $\mathbb{S}^3(\kappa,\tau)$. This shows $s_0 = d^*$ and (i) is proved.

The proof of (ii) is more delicate. We essentially follow the proof of [**Min93b**, Theorem 1], which is a generalisation of the application in [**Min93a**], and adjust it to our setup in $\mathbb{S}^3(\kappa,\tau)$ whenever needed. Without loss of generality (otherwise we apply an isometry) we may assume that h_1 is the horizontal geodesic

$$h_1\colon [0, 4\pi/\sqrt{\kappa}] \to \mathbb{S}^3(\kappa,\tau), \qquad h_1(t) = \begin{pmatrix} \cos\left(\frac{\sqrt{\kappa}}{2}t\right) \\ \sin\left(\frac{\sqrt{\kappa}}{2}t\right) \end{pmatrix}.$$

A minimal disc Σ_1 bounded by h_1 is parametrised by

$$[0, 4\pi/\sqrt{\kappa}] \times [0, 2\tau\pi/\kappa] \to \mathbb{S}^3(\kappa,\tau), \qquad (x,y) \mapsto \begin{pmatrix} \cos\left(\frac{\sqrt{\kappa}}{2}x\right) \\ \sin\left(\frac{\sqrt{\kappa}}{2}x\right)\exp(i\frac{\kappa}{4\tau}y) \end{pmatrix}.$$

We need a one-parameter family of minimal surfaces bounded by h_1 in order to construct a suitable family $A \subset X \times (0,1)$. Left-translation along h_1 defines a one-parameter family of isometries:

$$\Phi_t := \mathcal{L}_{h_1(t)}\colon \mathbb{S}^3(\kappa,\tau) \to \mathbb{S}^3(\kappa,\tau),\ \Phi_t(z,w) = \begin{pmatrix} \cos\left(\frac{\sqrt{\kappa}}{2}t\right)z - \sin\left(\frac{\sqrt{\kappa}}{2}t\right)\overline{w} \\ \cos\left(\frac{\sqrt{\kappa}}{2}t\right)w + \sin\left(\frac{\sqrt{\kappa}}{2}t\right)\overline{z} \end{pmatrix}.$$

We note that Σ_1 is not invariant by any Φ_t for $t \in (0, 2\pi/\sqrt{\kappa})$. Thus $\Sigma_1^t := \Phi_t(\Sigma_1)$ for $t \in I := [0, 4\pi/\sqrt{\kappa}]$ defines a one-parameter family of minimal surfaces bounded by h_1 such that $\Phi_0(\Sigma_1) = \Phi_{2\pi/\sqrt{\kappa}}(\Sigma_1)$.

Now let Σ_2 be a minimal surface bounded by h_2. We reason as in the proof of Proposition 5.1 to construct a one-parameter family of annuli $A \subset X \times (0,1)$ with the

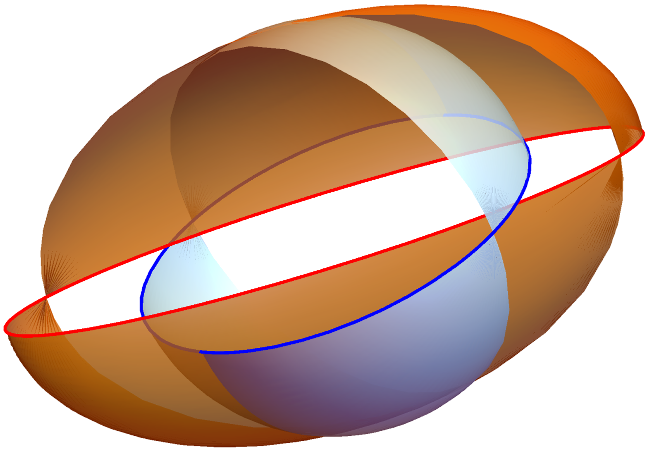


FIGURE 5.3. Stereographic projection of Σ_1^t and Σ_2 for a fixed $t \in I$; coloured in red and dark blue are the stereographic projections of the linked boundary curves

desired properties. We consider p^t and a neighbourhood $N = N(p^t)$ such that $\Sigma_1^t \cap N$ and $\Sigma_2 \cap N$ intersect transversally. Furthermore there is a diffeomorphism Φ_t from N into an open subset $U \subset \mathbb{R}^3$ such that Φ_t is an isometry at p^t and Φ_t maps $\Sigma_1^t \cap N$ respectively $\Sigma_2 \cap N$ into planes. The construction involving the prism then results in an annulus

$$\Sigma^t = \left(\Sigma_1^t \setminus \Phi_t^{-1}(R_1)\right) \cup \left(\Sigma_2 \setminus \Phi_t^{-1}(R_2)\right) \cup \Phi_t^{-1}(R)$$

such that $\operatorname{area}(\Sigma^t) < d^*$ for all $t \in I$. This construction depends continuously on t.

Let $\rho_t \in (0,1)$ and $u_t \colon A_{\rho_t} \to \Sigma^t$ be a conformal parametrisation of Σ^t such that u_t depends continuously on t. Let $v_t \in X$ be such that $v_t^{\rho_t} = u_t$. Then we obtain a continuous map

$$\mathcal{G} \colon I \to X \times (0,1), \qquad \mathcal{G}(t) := (v_t, \rho_t).$$

We will show that $A := \mathcal{G}(I)$ satisfies $\sup_A F < d^*$ and $i(A) \geq 2$. Since $\mathcal{G}$ is continuous and I is compact we get

$$\sup_{(u,\rho)\in A} F(u,\rho) = \max_{(u,\rho)\in A} F(u,\rho) < d^*.$$

We argue by contradiction to show $i(A) \geq 2$. Suppose that $\mathcal{H}\colon A \times [0,1] \to Z \times (0,1)$ deforms A to $[u^*] \times \{\rho^*\}$ for some $(u^*, \rho^*) \in Z \times (0,1)$. We can assume u^* is C^1 since Z is path-connected. The last claim follows from [**BL01**, Theorem 0.1], which states that $W^{1,p}(M_1, M_2)$ is path-connected for $p \geq \dim(M_1)$ if and only if $C^0(M_1, M_2)$ is path-connected. For $M_1 = \overline{A_\rho}$ and $M_2 = \mathbb{S}^3(\kappa, \tau)$ the set of continuous functions is path-connected since M_2 is simply connected and M_1 is compact.

We define a continuous map $\mathcal{F}\colon I \times [0,1] \to Z \times (0,1)$ by

$$\mathcal{F}(t,s) = \left(\mathcal{F}^1(t,s), \mathcal{F}^2(t,s)\right) := \mathcal{H}(\mathcal{G}(t), s), \qquad (t,s) \in I \times [0,1].$$

Notice that $\mathcal{F}^1$ is a map into Z. Therefore

$$f\colon A_{\rho_0} \times I \times [0,1] \to \mathbb{S}^3(\kappa,\tau), \qquad f(z,t,s) := \mathcal{F}^1(t,s)(z)$$

is continuous. The following properties are immediate by definition of f:

(a) $f(z,t,0) = v_t(z)$ for all $z \in A_{\rho_0}$ and all $t \in I$,
(b) $f(A_{\rho_0}, I, 1) \subset u^*(A_{\rho_0})$,
(c) $f(\partial A_{\rho_0}, I, s) \subset h_1 \cup h_2$ for all $s \in [0,1]$.

Property (c) implies that for all $Q \in \mathbb{S}^3(\kappa,\tau) \setminus (h_1 \cup h_2)$ the degree

$$[0,1] \to \mathbb{R}, \qquad s \mapsto \deg\left(f(\cdot, s), A_{\rho_0} \times I, Q\right)$$

is well-defined and continuous. Property (b) yields

$$\deg\left(f(\cdot,1), A_{\rho_0} \times I, Q\right) = 0 \quad \text{for all } Q \in \mathbb{S}^3(\kappa,\tau) \setminus u^*\left(\overline{A_{\rho_0}}\right).$$

However, we will show there exists a point $Q^* \in \mathbb{S}^3(\kappa,\tau) \setminus u^*\left(\overline{A_{\rho_0}}\right)$ such that

$$\deg\left(f(\cdot,0), A_{\rho_0} \times I, Q^*\right) = \pm 1.$$

This would be a contradiction since $f(\cdot,0)$ and $f(\cdot,1)$ are homotopy equivalent. By property (a) we only need to find a point $Q^* \in \mathbb{S}^3(\kappa,\tau) \setminus u^*\left(\overline{A_{\rho_0}}\right)$ such that

$$\text{there is a unique } (z^*, t^*) \in A_{\rho_0} \times I \text{ with } v_{t^*}(z^*) = Q^*.$$

From the construction we see that the prisms define a local foliation of a domain in $\mathbb{S}^3$; identifying the images of v^t with the map v^t itself, we see that the leafs are defined by restrictions of the maps v^t. Since u^* is assumed to be C^1, its image $u^*\left(\overline{A_{\rho_0}}\right)$ has measure zero in $\mathbb{S}^3$. Thus there is a leaf v^{t^*} and a desired point Q^* on that leaf, which finishes the proof. $\square$

Remark 5.6. We conclude this chapter with a remark on the proof of Theorem 5.5. Finding a set A that only satisfies $i(A) \geq 2$ is easier since the index is purely topological. For instance, we can deform $\mathbb{S}^3(\kappa, \tau)$ to $\mathbb{S}^3$ and h_1 and h_2 by an isotopy so that we are in the situation of a $\boldsymbol{j}$ and $(-\boldsymbol{j})$-Hopf circle in $\mathbb{S}^3$. Then we consider the family of minimal annuli corresponding to tilted unduloids in $\mathbb{R}^3$. This family is a foliation of $\mathbb{S}^3$ except for two great circles and the index is obviously greater or equal to 2. Going back with the isotopy we obtain a family A in X with $i(A) \geq 2$. However, we do not know whether each element of this family satisfies the Douglas condition. The proof above can be considered a "local" version of this argument.

CHAPTER 6

On the existence problem for tilted unduloids in $\mathbb{H}^2 \times \mathbb{R}$

In the present chapter we study the existence problem for singly periodic (Alexandrov) embedded MCH-annuli in $\mathbb{H}^2 \times \mathbb{R}$ with a *tilted axis*, that is, an axis neither vertical nor horizontal. We refer to these surfaces as *tilted unduloids*. In Theorem 2.7 we have seen that such surfaces exist. However, these surfaces are invariant under a one-parameter family of translations. Do singly periodic (Alexandrov) embedded MCH-annuli in $\mathbb{H}^2 \times \mathbb{R}$ with a *tilted axis* exist that are only invariant under a discrete group of translations?

We propose the following conjugate Plateau construction to answer this question:

- Choose horizontal geodesics h_1 and h_2 in $\mathbb{S}^3(4H^2-1, H)$ and fix their orientations. Construct a minimal annulus Σ_0 bounded by h_1 and h_2 for the given orientation of h_1 and h_2 and consider the universal cover Σ of Σ_0.
- Daniel's correspondence yields an MCH-surface $\widetilde{\Sigma}$ in $\mathbb{H}^2 \times \mathbb{R}$.
- Extend $\widetilde{\Sigma}$ to a tilted unduloid by reflection.

The task is to impose conditions on h_1 and h_2, as well as Σ, such that $\widetilde{\Sigma}$ extends to a tilted unduloid. It will turn out that with this approach we can reduce the existence problem to a uniqueness problem for minimal annuli bounded by horizontal geodesics in the Berger spheres. Currently we are not able to assert the uniqueness.

In Section 6.1 we formulate three hypotheses (H1) to (H3) for the universal cover Σ of a minimal annulus bounded by linked horizontal geodesics h_1 and h_2. These hypotheses concern the MCH-sister $\widetilde{\Sigma}$ or the sister curves $\widetilde{h_1}$ and $\widetilde{h_2}$. If Σ satisfies (H1) to (H3) and the horizontal fields of h_1 and h_2 are linearly independent we show in Theorem 6.1 that $\widetilde{\Sigma}$ extends to a tilted unduloid in $\mathbb{H}^2 \times \mathbb{R}$.

The hypotheses (H1) to (H3) are, at first sight, strong assumptions on the geometry of the sister surface respectively the sister curves. We devote Section 6.2 to suggest how they could be verified with our approach. Given a pair of linked horizontal geodesics h_1 and h_2 as above we believe and conjecture that there are exactly two minimal annuli bounded by h_1 and h_2 in $\mathbb{S}^3(4H^2-1, H)$. A consequence of such a uniqueness theorem would be continuity of the solutions depending on h_1 and h_2. Starting with a solution that satisfies (H1) to (H3) (for example a horizontal unduloid or a tilted MCH-cylinder) we see that, under a slight deformation of h_1 and h_2, the hypotheses (H1) and (H3) follow from this continuity property. We conclude with a conjecture how we expect tilted unduloids to arise in terms of h_1 and h_2. In Section 6.3 we finish this chapter with a remark on the construction of nodoids.

6.1. Conjugate construction of tilted unduloids in $\mathbb{H}^2 \times \mathbb{R}$ under hypotheses

It is useful to introduce the following notation for the conjugate Plateau construction outlined in the introduction to this chapter:

Notation. *Let h_1 and h_2 be linked horizontal geodesics in $\mathbb{S}^3(\kappa, \tau)$ with prescribed orientations. We set*

$$\mathcal{M}_0 := \mathcal{M}_0(h_1, h_2) := \{\Sigma_0 \colon \Sigma_0 \textit{ is a minimal annulus bounded by } h_1 \textit{ and } h_2\}$$

and

$$\mathcal{M} := \mathcal{M}(h_1, h_2) := \{\Sigma \colon \Sigma \textit{ is the minimal universal cover of } \Sigma_0 \in \mathcal{M}_0\}\,.$$

We finally pose the existence question: Do tilted unduloids in $\mathbb{H}^2 \times \mathbb{R}$ exist? An affirmative answer to our problem depends on the following hypotheses whose geometric meaning is explained in Theorem 6.1:

Definition (Hypotheses)**.** Let $H > \frac{1}{2}$ and let h_1 and h_2 be linked horizontal geodesics with prescribed orientations in $\mathbb{S}^3(4H^2-1, H)$, having length L. Let $\Sigma \in \mathcal{M}(h_1, h_2)$ and consider the MCH-sister surface $\widetilde{\Sigma}$ in $\mathbb{H}^2 \times \mathbb{R}$. By P_j we denote the vertical plane containing the sister curve $\widetilde{h}_j$, where $j \in \{1, 2\}$. The minimal surface Σ *satisfies (H1) to (H3)* if the following is true:

(H1) Either there exists $j \in \{1, 2\}$ such that $\widetilde{h}_j(0) \neq \widetilde{h}_j(L)$ or if $\widetilde{h}_j(0) = \widetilde{h}_j(L)$ for all $j \in \{1, 2\}$ then $P_1 \cap P_2 = \emptyset$.

(H2) The MCH-surface $\widetilde{\Sigma}$ has a non-vertical axis.

(H3) If $\widetilde{\Sigma}$ extends to an MCH-annulus by reflections through P_1 and P_2 then the annulus is (Alexandrov) embedded.

We state the main result of this chapter:

Theorem 6.1. *Let $H > \frac{1}{2}$ and consider linked horizontal geodesics h_1 and h_2 with horizontal fields F_1 and F_2 in $\mathbb{S}^3(4H^2-1, H)$. Let $\Sigma \in \mathcal{M}(h_1, h_2)$.*

- *(i) Assume the MCH-sister surface $\widetilde{\Sigma}$ satisfies (H1). Then $\widetilde{\Sigma}$ is a singly periodic MCH-surface in $\mathbb{H}^2 \times \mathbb{R}$.*
- *(ii) If F_1 and F_2 are linearly independent and $\widetilde{\Sigma}$ satisfies (H1) to (H3) then $\widetilde{\Sigma}$ extends to a tilted unduloid in $\mathbb{H}^2 \times \mathbb{R}$.*

Proof. We prove (i). Let $L := 4\pi/\sqrt{4H^2-1}$ be the length of a horizontal geodesic in $\mathbb{S}^3(4H^2-1, H)$. Then we have $h_1(0) = h_1(L)$ and $h_2(0) = h_2(L)$. When applying the conjugate sister relation, the cases

$$\text{(A)} \quad \widetilde{h_1}(0) \neq \widetilde{h_1}(L) \qquad \text{or} \qquad \text{(B)} \quad \widetilde{h_1}(0) = \widetilde{h_1}(L). \tag{6.1}$$

can occur.

Let us assume (A) first. Then $p := \widetilde{h_1}(0)$ and $q := \widetilde{h_1}(L)$ are distinct points in the same vertical plane P_1. There is an isometric translation Φ, acting without fix points, such that $\Phi(p) = q$. We claim that the surface $\widetilde{\Sigma}$ is invariant under Φ and thus singly periodic. In order to show this we want to use the Fundamental Theorem of Surfaces in homogeneous 3-manifolds, see [**Dan07**, Sections 3 and 4].

We consider the annulus Σ_0 as a minimal immersion $f_0 \colon \Omega_0 \to \mathbb{S}^3(4H^2-1, H)$ defined on an annulus $\Omega_0 \subset \mathbb{R}^2$. The universal cover Σ of Σ_0 is then a minimal immersion $f \colon \Omega \to \mathbb{S}^3(4H^2-1, H)$ defined of the universal cover Ω of Ω_0. The geometric data are the first fundamental form, the shape operator S, the vertical component of the Gauß map $\langle Z, N\rangle$, and the vector field $df^{-1}(T)$ as in Proposition 3.8. The MCH-sister $\tilde{f}$ has the same first fundamental form (Daniel correspondence is isometric) and shape operator $\widetilde{S} = JS + H\,\mathrm{id}$. Furthermore we know $\langle \widetilde{Z}, \widetilde{N}\rangle = \langle Z, N\rangle$ and $df^{-1}(T) = Jd\tilde{f}^{-1}(\widetilde{T})$. Since Σ_0 is an annulus we see that the geometric data at the points p and q are equal. That is, periodic data of the minimal annulus Σ_0 in the Berger spheres imply periodic data for the MCH-sister surface $\widetilde{\Sigma}$. Integrating along curves $\tilde{c} = \tilde{f} \circ \gamma$ joining p and q, the Fundamental Theorem of Surfaces in homogeneous 3-manifolds implies that Φ generates the isometry group of the MCH-surface $\widetilde{\Sigma}$. In particular, $\widetilde{h_2}$ cannot be a closed curve.

Now we consider case (B). If $\widetilde{h_1}$ is closed then $\widetilde{h_2}$ is closed, too. Otherwise we can argue as in case (A) to show that the surface is singly periodic. We assume (H1), that is, the vertical planes P_1 and P_2 containing $\widetilde{h_1}$ and $\widetilde{h_2}$, respectively, are disjoint. Then we can reflect through P_1 and P_2 to obtain a singly periodic MCH-annulus with a horizontal axis.

Now we prove (ii). We claim that (H2) and (H3) imply case (A) in the proof of (i). Assume that we were in case (B). Then we can extend $\widetilde{\Sigma}$ by reflections through P_1 and P_2 to an MCH-annulus $\widetilde{\Sigma}$ with a horizontal axis. Hypothesis (H3) guarantees that $\widetilde{\Sigma}$ is (Alexandrov) embedded, hence it has a vertical mirror plane P by Proposition 3.14 (ii). Let $\widetilde{h}$ be a mirror curve in P, see also Figure 6.1.

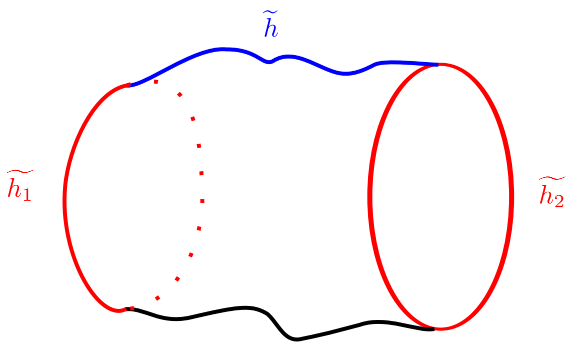


FIGURE 6.1. Case (B) from (6.1) is impossible in (ii)

Then the sister curve h is a horizontal geodesic that intersects h_1 and h_2 orthogonally. This implies $F_1 = \pm F_2$ for the horizontal fields F_1 and F_2, a contradiction to our assumption on these fields being linearly independent.

Thus $\widetilde{\Sigma}$ satisfies case (A) and is singly periodic. By (H2) the axis is non-vertical axis so that the vertical planes P_1 and P_2 must be equal by Proposition 3.13 (i). Hence reflection σ through $P_1 = P_2$ extends $\widetilde{\Sigma}$ to a properly immersed MCH-annulus $\widetilde{\Sigma} \cup \sigma(\widetilde{\Sigma})$. The singly periodic annulus $\widetilde{\Sigma} \cup \sigma(\widetilde{\Sigma})$ is (Alexandrov) embedded by (H3). The axis cannot be horizontal since Proposition 4.5 implied $F_1 = \pm F_2$, contradicting our assumption on F_1 and F_2. Thus the axis is tilted. □

6.2. Discussion of hypotheses in construction and conjecture on tilted unduloids

In Theorem 6.1 we have shown that tilted unduloids in $\mathbb{H}^2 \times \mathbb{R}$ exist if there are horizontal geodesics h_1 and h_2 with linearly independent horizontal fields F_1 and F_2

such that some $\Sigma \in \mathcal{M}(h_1, h_2)$ satisfies (H1) to (H3). These hypotheses can be verified if we assume the following conjecture to be true:

Conjecture 6.2 (Uniqueness)**.** *Let h_1 and h_2 be linked horizontal geodesics in a Berger sphere $\mathbb{S}^3(\kappa, \tau)$ with $\kappa \neq 4\tau^2$. Then we have $|\mathcal{M}_0(h_1, h_2)| = 2$, that is, there are exactly two minimal annuli bounded by h_1 and h_2.*

In the special case of $\mathbb{S}^3$, that is for $\kappa = 4\tau^2$, we observed in Theorem 4.8 that there is an orbit of minimal annuli bounded by a $\boldsymbol{j}$-Hopf circle and a $(-\boldsymbol{j})$-Hopf circle. This orbit is induced by a one-parameter family of isometries in $\mathbb{S}^3$. Deforming $\mathbb{S}^3$ to a Berger sphere, these are not isometries anymore. Nevertheless, in $\mathbb{S}^3(\kappa, \tau)$ we have at least two solutions: A "Min" and a "Minimax" for the Dirichlet energy. This follows from Theorem 5.5. We believe that only these two solutions preserve in the deformation from $\mathbb{S}^3$ to $\mathbb{S}^3(\kappa, \tau)$.

A similar uniqueness result is available in $\mathbb{R}^3$: In [**MW93a**, Theorem 1.2] Meeks and White showed that an extremal pair of smooth disjoint convex curves in distinct planes bounds at most two minimal annuli. The case of two minimal annuli is realised by two generic solutions, a "Min" and a "Minimax", even though the case of no solutions occurs.

We now assume Conjecture 6.2 to be true. For $c \in (0, 1]$ let f^c be the spherical helicoid from Theorem 4.1. Consider $h_1 := f^c(\cdot, 0)$ and $h_2^\alpha := f^c(\cdot, \alpha)$ for

$$\alpha \in \left(0, 8H \frac{\pi}{(c+1)(4H^2 - 1)}\right) =: I.$$

For fixed c, we obtain two one-parameter families of minimal annuli, namely, $(\Sigma_1^\alpha)_{\alpha \in I}$ corresponding to pieces of vertical unduloids, and another family $(\Sigma_2^\alpha)_{\alpha \in I}$. The assumption $|\mathcal{M}_0(h_1, h_2^\alpha)| = 2$ and Example 4.6 show that for

$$\alpha_0 = 4H \frac{\pi}{(c+1)(4H^2 - 1)}$$

the MCH-sister surface $\widetilde{\Sigma_2^{\alpha_0}}$ is one half of a horizontal unduloid in $\mathbb{H}^2 \times \mathbb{R}$.

Since $|\mathcal{M}_0(h_1, h_2^\alpha)| = 2$ for all $\alpha \in I$ we see that Σ_2^α depends continuously on α (we also need curvature estimates for minimal annuli to show this). The surface $\Sigma_2^{\alpha_0}$ satisfies (H1) to (H3): Indeed, let N^α and $\widetilde{N}^\alpha$ denote the normals to Σ_2^α and $\widetilde{\Sigma_2^\alpha}$, respectively. Then we have

$$\langle N^{\alpha_0} \circ h_1, \xi \circ h_1 \rangle = \left\langle \widetilde{N}^{\alpha_0} \circ \widetilde{h_1}, \tilde{\xi} \circ \widetilde{h_1} \right\rangle \neq 0 \tag{6.2}$$

since $c > 0$ and because the horizontal unduloid is a vertical bi-graph by the construction of Manzano and Torralbo; see [**MT14**] respectively Example 4.6 and note that $\widetilde{\gamma}_1$ in the illustration on page 60 extends to $\widetilde{h_1}$. The continuity property shows that for α close to α_0 the surface Σ_2^α has no branch points, i.e., it is an immersion, and (6.2) is preserved for α close to α_0. This shows three properties:

- $\widetilde{h_2}^\alpha(0) \neq \widetilde{h_2}^\alpha(L)$: otherwise the normal along the sister curve $\widetilde{h_2}^\alpha$ in the vertical plane P_2^α would be horizontal, a contradiction to the choice of α. Therefore $\widetilde{\Sigma_2^\alpha}$ is singly periodic as shown in Theorem 6.1 (i).
- The horizontal fields F_1 and F_2^α are linearly independent for all $\alpha \in I \setminus \{\alpha_0\}$ since $F_2^{\alpha_0} = -F_1$ by definition of α_0. We refer to Theorem 4.1 (i) for an explicit computation of the horizontal field.
- The axis is non-vertical, i.e., (H2) is satisfied: If $\widetilde{\Sigma_2^\alpha}$ had a vertical axis then the normal along $\widetilde{h_1}$ would be horizontal at some point, contradicting the choice of α once again.

Hypothesis (H3) follows since (Alexandrov) embeddedness is preserved under continuous deformations. Such a deformation argument has been used to show that minimal spheres in a compact homogeneous 3-manifold are Alexandrov embedded, see [**MP12**, Lemma 4.3 and Corollary 4.4].

This shows the existence of $\varepsilon \in (0, \alpha_0)$ such that for each $\alpha \in (\alpha_0 - \varepsilon, \alpha_0 + \varepsilon) \setminus \{\alpha_0\}$ there is $\Sigma^\alpha \in \mathcal{M}(h_1, h_2^\alpha)$ so that $\widetilde{\Sigma^\alpha}$ extends to a tilted unduloid in $\mathbb{H}^2 \times \mathbb{R}$. We believe this conclusion to be true for all $\alpha \in I \setminus \{\alpha_0\}$ and formulate the conjecture on tilted unduloids as follows:

Conjecture 6.3 (Tilted unduloid)**.** *Let $H > \frac{1}{2}$ and $c \in (0, 1]$. We consider the spherical helicoid f^c as in Theorem 4.1. Let $h_1 := f^c(\cdot, 0)$ and $h_2^\alpha := f^c(\cdot, \alpha)$ for $\alpha \in I$. Then for each $\alpha \in I \setminus \{\alpha_0\}$ there is $\Sigma^\alpha \in \mathcal{M}(h_1, h_2^\alpha)$ so that $\widetilde{\Sigma^\alpha}$ extends to a tilted unduloid in $\mathbb{H}^2 \times \mathbb{R}$.*

The geometry is not completely understood. Namely, we cannot determine the exact slope of the axis or the neck size of a tilted unduloid.

We recall that for the family of spherical helicoids f^c the parameter $c \in (0, 1]$ corresponds to the neck size of the vertical unduloid in $\mathbb{H}^2 \times \mathbb{R}$. If we do not fix c in the construction above, we obtain two two-parameter families of universal covers of minimal annuli

$$(\Sigma_1^{\alpha,c})_{(\alpha,c)\in I\times(0,1)} \quad \text{and} \quad (\Sigma_2^{\alpha,c})_{(\alpha,c)\in I\times(0,1)}\,.$$

In the first family α parametrises what piece of a vertical unduloid we consider and c is the neck size of the vertical unduloid. For the second family the following natural question arises, provided that the conjectures on the uniqueness of minimal annuli and tilted unduloids are true:

Question. *Does α parametrise the slope of the axis of $\widetilde{\Sigma_2^{\alpha,c}}$? Is c also the neck size in this family?*

In order to answer this question and to make the conjecture on tilted unduloids more precise it would be useful to have the sister curves h_1 and h_2 of a tilted MCH-cylinder in $\mathbb{H}^2\times\mathbb{R}$. In the Outlook we outline a construction of surfaces in the Berger spheres whose MCH-sisters we believe to be tilted MCH-cylinders.

6.3. Remark on construction of nodoids

The construction outlined in Section 6.1 yields existence of tilted unduloids if hypotheses (H1) to (H3) are satisfied. If they are not satisfied then we still obtain MCH-surfaces in $\mathbb{H}^2\times\mathbb{R}$. We believe this surfaces might be nodoids in $\mathbb{H}^2\times\mathbb{R}$.

A nodoid in $\mathbb{R}^3$ is a rotationally invariant MCH-surface generated by the roulette of a hyperbola. Therefore a nodoid is a singly periodic MCH-annulus that is not (Alexandrov) embedded. In $\mathbb{H}^2\times\mathbb{R}$ there exist vertical nodoids; they are the MCH-sister surfaces of the spherical helicoids f^c from Theorem 4.1 for $c\in(-1,0)$. This corresponds to a different orientation of the boundary curves. It should be possible to construct horizontal nodoids à la Manzano and Torralbo in [**MT14**].

Conjecture 6.4. *There exist* horizontal nodoids *in $\mathbb{H}^2\times\mathbb{R}$; they have constant mean curvature $H>\frac{1}{2}$ and are singly periodic with respect to hyperbolic translations.*

We believe nodoids exist with respect to any axis in $\mathbb{H}^2\times\mathbb{R}$. The arguments used in Section 6.1 and Section 6.2 to construct tilted unduloids require (Alexandrov) embeddedness which fails for the case of a tilted nodoid. Once we have tilted unduloids as in Conjecture 6.3, we believe tilted nodoids to arise by a change of orientations of the boundary curves h_1 and h_2^α. This requires a deeper study.

Outlook, Appendices and Backmatter

Outlook

In this chapter we discuss problems or approaches that are directly or closely related to results obtained in this thesis, but could not be studied in detail while finishing this thesis.

Minimal sisters of tilted MCH-cylinders in $\mathbb{H}^2 \times \mathbb{R}$

In Chapter 4 we considered only vertical and horizontal unduloids in $\mathbb{H}^2 \times \mathbb{R}$ in order to obtain information on the horizontal geodesics in $\mathbb{S}^3(4H^2-1, H)$. It would be a nice result to have the sister curves of a tilted MCH-cylinder in $\mathbb{H}^2 \times \mathbb{R}$ explicitly in $\mathbb{S}^3(4H^2-1, 4H)$. We can assume one of the horizontal geodesics to be

$$h(s) = \begin{pmatrix} \cos\left(\frac{\sqrt{k}}{2}s\right) \\ \sin\left(\frac{\sqrt{k}}{2}s\right) \end{pmatrix}.$$

The one-parameter family of isometries $\left(\mathcal{L}_{h(s)}\right)_{s\in\mathbb{R}}$ defines an invariant surface

$$f(s,t) := \mathcal{L}_{h(s)}(z(t), w(t)),$$

where $t \mapsto (z(t), w(t))$ is some C^2-curve in $\mathbb{S}^3(4H^2-1, H)$. If we choose $(z(t), w(t))$ to be a vertical or horizontal geodesic orthogonal to h at $h(0)$ then the invariant surface is a minimal surface that corresponds to a vertical cylinder or to a horizontal cylinder in $\mathbb{H}^2 \times \mathbb{R}$.

For a general curve $(z(t), w(t))$ requiring the invariant surface f to be minimal leads to an ODE for $(z(t), w(t))$. Choosing the initial values carefully we can guarantee that the normal N to f satisfies $\langle N \circ h, \xi \circ h\rangle = \cos(\alpha)$ for some $\alpha \in (0, \pi/2)$. Arguing as in Theorem 6.1 (i) we see that the MCH-sister $\tilde{f}$ is a translationally invariant MCH-surface as in Chapter 2. However, we do not know whether the sister curve of $(z(t), w(t))$ is contained in a vertical plane or not. We also do not know if there is t_0 such that $f(\cdot, t_0)$ is a horizontal geodesic different from h. We believe this information can be obtained by a flux computation as in Section 2.3.

Minimax principle for minimal surfaces with higher connectivity

The minimax principle by [**Min93a**] should also be true for k-circle domains, that is, for a disc D from which $k-1$ closed discs inside D are removed and which do not intersect. In that sense a 1-circle domain is a disc and a 2-circle domain is an annulus. The main problems regarding the analysis arise in the cases $k=1$ and $k=2$, which have been dealt with already. Since Jost proved the generalised Douglas problem in Riemannian 3-manifolds (see [**Jos91**]) a generalisation to k-circle domains could only be problematic with respect to the index i. Also, for the same type of domains a similar multiple solution theorem has been obtained for the Dirichlet problem in $\mathbb{S}^n$ with respect to such a k-circle domain; see [**Din85**, Theorem 4.6].

Conjecture. *There is a minimax principle for minimal surfaces of k-circle domain type.*

Intermezzo: Constant mean curvature surfaces in metric Lie groups

The following ideas rely on a generalised Weierstrass representation of MCH-surfaces in metric Lie groups and thus we will first introduce some notation.

Let $(X, \langle\cdot,\cdot\rangle)$ be a *metric Lie group*, that is, X is a Lie group and $\langle\cdot,\cdot\rangle$ a Riemannian metric on X such that left-translations are isometries. The $E(\kappa,\tau)$-spaces except for $\mathbb{S}^2\times\mathbb{R}$ and Sol_3 are examples. According to [**MMPR13**, Section 2] the metric Lie group X can either be *unimodular* or *non-unimodular*.

If X is unimodular then there exists a left-invariant orthonormal frame (E_1, E_2, E_3) and $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$[E_2, E_3] = c_1 E_1, \quad [E_3, E_1] = c_2 E_2, \quad [E_1, E_2] = c_3 E_3.$$

Associated to c_1, c_2 and c_3 are the numbers

$$\mu_1 = \frac{1}{2}\left(-c_1 + c_2 + c_3\right), \quad \mu_2 = \frac{1}{2}\left(c_1 - c_2 + c_3\right), \quad \mu_3 = \frac{1}{2}\left(c_1 + c_2 - c_3\right).$$

The Berger spheres with its orthonormal frame introduced in Section 3.1 are an example of a compact unimodular group. The space Sol_3 is also unimodular: Consider the frame

(E_1, E_2, E_3) introduced in Section 1.1. Then the frame $\left(-\frac{E_1+E_2}{\sqrt{2}}, \frac{E_1-E_2}{\sqrt{2}}, E_3\right)$ satisfies the relations above for the values $c_1 = 1$, $c_2 = -1$ and $c_3 = 0$.

If X is non-unimodular then $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ for a matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a + d \neq 0$. This generalises the model used for the $E(\kappa, \tau)$-spaces with $\kappa \leq 0$ in Section 2.2. One can rescale the matrix A and thus the metric on $\mathbb{R}^2 \rtimes_A \mathbb{R}$ such that

$$A(a, b) = \begin{pmatrix} 1+a & -(1-a)b \\ (1+a)b & 1-a \end{pmatrix}, \quad a, b \in [0, \infty).$$

Definition (Gauß map and H-potential)**.** Let X be a metric Lie group.

(i) Let $f\colon \Sigma \to X$ be an oriented immersed surface with unit normal vector field $N\colon \Sigma \to TX$, we define the *left-invariant Gauß map* of the immersed surface to be the map $G\colon \Sigma \to \mathbb{S}^2 \subset T_eX$ that assigns to each $p \in \Sigma$ the unit tangent vector to X at the identity element e given by $(d\mathcal{L}_{f(p)})_e(G(p)) = N_p$.

(ii) Given $H \in \mathbb{R}$, we define the *H-potential* of X to be the map $R\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ by

$$R(q) = H\left(1+|q|^2\right)^2 - \left(1-|q|^4\right) - a\left(q^2 - \overline{q}^2\right) - ib(2|q|^2 - a(q^2 + \overline{q}^2)),$$

if $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ is non-unimodular and A normalised as above, or by

$$R(q) = H\left(1+|q|^2\right)^2 - \frac{i}{2}\left(\mu_2|1+q^2|^2 + \mu_1|1-q^2|^2 + 4\mu_3|q|^2\right)$$

if X is unimodular.

Given an oriented immersed surface $f\colon \Sigma \to X$ with Gauß map G and canonical orthonormal frame (E_1, E_2, E_3) we can consider the stereographic projection of G with respect to the south pole $-E_3$ in $\mathbb{S}^2 \subset T_eX$. We thus obtain a map $g\colon \Sigma \to \overline{\mathbb{C}}$. If f is a conformal MCH-immersion this map g satisfies the complex elliptic PDE

$$g_{z\overline{z}} = \frac{R_q}{R}(g) g_z g_{\overline{z}} + \left(\frac{R_{\overline{q}}}{R} - \frac{\overline{R_q}}{\overline{R}}\right)(g)|g_z|^2. \tag{6.3}$$

In [**MMPR13**, Theorem 3.7] this equation is shown to be sufficient for a conformal MCH-immersion $f\colon \Sigma \to X$ with Gauß map g to exist. Thus we can construct MCH-surfaces in X by finding a suitable Gauß map. The normal N of the surface is explicitly defined in terms of g:

$$N = \frac{1}{1+|g|^2}\left((g+\overline{g})E_1 + i(\overline{g} - g)E_2 + (1-|g|^2)E_3\right).$$

An explicit parametrisation has to be integrated, which is not always possible.

Interesting examples have been constructed by exhibiting Gauß maps g solving (6.3). For example, horizontal helicoids and horizontal catenoids in Heisenberg space by Daniel and Hauswirth in [**DH09**]; this approach has been carried over to Sol_3 by Desmonts in [**Des15**] to construct helicoids and catenoids in Sol_3. Also invariant surfaces can be analysed in terms of its Gauß map, see for instance [**DM13**, Example 3.9] for some invariant MCH-surfaces in Sol_3.

Problem 1: Gauß maps of MCH-cylinders

In the first part of this thesis we constructed translationally invariant MCH-cylinders in Sol_3 and non-compact $E(\kappa,\tau)$-spaces with a geometric approach. By [**MP12**, Corollary 3.8] the Gauß map g of these MCH-cylinders is a regular curve in $\overline{\mathbb{C}}$. Since the normal along a generating curve is periodic the Gauß map g should be a periodic/closed regular curve. The ansatz $g(z,\overline{z}) = r(z+\overline{z})\exp(i(z+\overline{z}))$ for a real-valued function r results in an ODE for r that can be studied. Especially in Sol_3 it could be useful to try this approach regarding the non-embedded MCH-cylinders with axis c.

Problem 2: Tilted unduloids in $\mathbb{H}^2\times\mathbb{R}$ via Gauß map

In addition to the geometric approach outlined in Section 6.1 and Section 6.2 it is also useful to have more explicit approaches at hand. An advantage of the Gauß map is the fact that properties of the normal are explicit.

Minimal sisters of tilted MCH-cylinders in $\mathbb{H}^2\times\mathbb{R}$. At the beginning of this outlook we have sketched how we expect the minimal sisters of tilted MCH-cylinders to arise. We believe that they are minimal surfaces invariant under left-translations along the horizontal geodesic

$$h(s) = \begin{pmatrix} \cos\left(\frac{\sqrt{k}}{2}s\right) \\ \sin\left(\frac{\sqrt{k}}{2}s\right) \end{pmatrix}.$$

The Gauß map g of such a minimal surface is therefore a regular curve in $\overline{\mathbb{C}}$. Is it possible to find this Gauß map as solution of an ODE?

Deformation of cylinder solution. Let $g(z,\overline{z}) = \gamma(z+\overline{z})$ be the Gauß map of the MCH-cylinder solution and consider $g_\varphi(z,\overline{z}) = \varphi(z-\overline{z})\gamma(z+\overline{z})$. Requiring g_φ to satisfy (6.3) leads to an ODE for φ in terms of the (not explicitly known) cylinder solution γ.

Question. *Can one deform g via φ to construct a singly periodic solution?*

One can try this numerically, for example. One should try this approach first in $\mathbb{R}^3$ in order to construct unduloids there.

A minimal surface in the Berger spheres by de Lira and Hinojosa. De Lira and Hinojosa construct a minimal surface in [**dLH10**, Section 6.4]. They exhibit an explicit Gauß map g in $\mathbb{S}^3(\kappa, \tau)$ that solves (6.3) and describe it as *ruled minimal surface*. The Gauß map seems similar to that of the helicoids in Heisenberg space or Sol_3, see [**DH09**] and [**Des15**].

The map g depends on τ, that is, it is a minimal surface that is not minimal in all Berger spheres. Thus it is not congruent to a spherical helicoid, which are the only surfaces that are minimal in all Berger spheres. The Gauß map is also not a curve, so that it is not an invariant minimal surface.

Question. *What kind of minimal surface is it? What is the sister surface in $\mathbb{H}^2 \times \mathbb{R}$?*

Problem 3: Singly periodic MCH-annuli in Sol_3

In the second part of the thesis we restricted the ambient space to $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{R}^3$. The reason for this are the Daniel and Lawson correspondence, respectively. Such a correspondence is not available for Sol_3.

Lawson and Daniel correspondence in terms of Gauß map. The Lawson correspondence for MCH-surfaces in $\mathbb{R}^3$ and minimal surfaces in $\mathbb{S}^3$ implies that the normals of the respective surfaces are equal. This is easy to see by looking at (6.3): The equation is the same for MCH-surfaces in $\mathbb{R}^3$ and minimal surfaces in $\mathbb{S}^3$. We note that both $\mathbb{R}^3$ and $\mathbb{S}^3$ are unimodular.

Question. *Is it possible to prove Proposition 3.11 (relation of sister curves) only in terms of the Gauß map g?*

It is also interesting to study the Daniel correspondence. There we only know that the E_3-component of the normal is preserved. We have already seen that Daniel's correspondence is weaker than the Lawson correspondence. Is this related to $\mathbb{H}^2 \times \mathbb{R}$ being a non-unimodular Lie group and the Berger spheres being a unimodular Lie group?

Speculation on correspondence for MCH-surfaces in Sol_3. A compact homogeneous 3-manifold with 3-dimensional isometry group can be obtained as follows: Start with $\mathbb{S}^3$ and its canonical orthonormal frame (E_1, E_2, E_3). Then (aE_1, bE_2, cE_3) for $a, b, c > 0$ all distinct defines a metric on $\mathbb{S}^3$ such that $\mathbb{S}^3$ is a unimodular Lie group X with 3-dimensional isometry group. The Berger spheres have been obtained in this way by setting $a = b \neq c$.

Question. *Is it possible to choose $a, b, c > 0$ such that minimal surfaces in this sphere X correspond to* MC*H-surfaces in Sol_3? If so, do we have a relation of sister curves?*

Deformation of cylinder solution in Sol_3 and $\mathbb{R}^3$. This is the same idea as in the case of $\mathbb{H}^2 \times \mathbb{R}$, see above.

APPENDIX A

ODE for $\textsc{mc}H$-cylinders in Sol_3 with axis c

We compute the ODE for $\textsc{mc}H$-cylinders with axis c in Sol_3 and include the basic Mathematica lines used to compute the examples in Section 1.2.

A.1. Computation of ODE

The mean curvature of an surface f invariant by translations along the base in Sol_3 is easy to compute in terms of the orthonormal frame from Section 1.1:

Proposition A.1. *Let f be as in* (1.3)*, i.e., f parametrises a surface invariant by translations along the base c in Sol_3. Then we have*

$$C := \sqrt{\det(g)} = \sqrt{x'^2 + y'^2 + (x'y + xy')^2}$$

for the induced Riemannian metric g on $\mathbb{R} \times J$. Moreover the mean curvature H of f in terms of $\gamma = (x, y, 0)$ with respect to the inner normal satisfies the equation

$$\begin{aligned} 2HC^3 &= \left[xy' - x'y + (x^2 - y^2)(xy' + x'y)\right] \cdot \left[x'^2 + y'^2\right] \\ &\quad + 2(yy' + xx')(yy' - xx')(xy' + x'y) \\ &\quad + (x^2 + y^2 + 1)\left(x'y'' - x''y' + (x'^2 - y'^2)(xy' + x'y)\right). \end{aligned} \tag{A.1}$$

Sketch of proof. We have

$$v_1 := \frac{\partial f}{\partial s}(s,t) = \begin{pmatrix} -e^{-s}x(t) \\ e^s y(t) \\ 1 \end{pmatrix} = -xE_1 + yE_2 + E_3$$

and

$$v_2 := \frac{\partial f}{\partial t}(s,t) = \begin{pmatrix} e^{-s}x'(t) \\ e^s y'(t) \\ 0 \end{pmatrix} = x'E_1 + y'E_2.$$

Thus the upper normal N to f is

$$N = \frac{1}{\sqrt{x'^2 + y'^2 + (x'y + xy')^2}} \left[-y'E_1 + x'E_2 - (xy' + x'y)E_3\right]. \tag{A.2}$$

The entries of the induced metric $g = (\langle v_j, v_k \rangle)_{1 \leq j,k \leq 2}$ on $\mathbb{R} \times J$ are

$$\begin{aligned} g_{11} &= x^2 + y^2 + 1, \\ g_{12} &= -xx' + yy', \\ g_{22} &= x'^2 + y'^2. \end{aligned} \tag{A.3}$$

Furthermore let us compute $\nabla_{v_j} v_k$ for $j, k \in \{1, 2\}$:

$$\begin{aligned} \nabla_{v_1} v_1 &= -xE_1 - yE_2 + (y^2 - x^2)E_3, \\ \nabla_{v_1} v_2 &= (xx' + yy')E_3, \\ \nabla_{v_2} v_2 &= x''E_1 + y''E_2 + \left(y'^2 - x'^2\right) E_3. \end{aligned}$$

It can be checked that $C := \sqrt{\det(g)}$ agrees with the denominator of the coefficients in (A.2), i.e. we have

$$C \cdot N = -y'E_1 + x'E_2 - (xy' + x'y)E_3.$$

Thus the second fundamental form $b = \left(\langle \nabla_{v_j} v_k, N \rangle\right)_{1 \leq j,k \leq 2}$ satisfies:

$$\begin{aligned} Cb_{11} &= xy' - x'y + (x^2 - y^2)(xy' + x'y), \\ Cb_{12} &= -(xx' + yy')(xy' + x'y), \\ Cb_{22} &= -x''y' + x'y'' + (x'^2 - y'^2)(xy' + x'y). \end{aligned}$$

In order to verify (A.1), the previous expressions must be plugged into

$$2H = \frac{g_{22}}{C^2} b_{11} - 2\frac{g_{12}}{C^2} b_{12} + \frac{g_{11}}{C^2} b_{22} = \frac{g_{22}Cb_{11} - 2g_{12}Cb_{12} + g_{11}Cb_{22}}{C^3}. \qquad \square$$

A.2. Mathematica code

In Section 1.2 we considered the following initial value problem for the generating curve $\gamma = (x, y, 0)$:

$$\gamma \text{ solves (A.1)} \quad \text{with } \gamma(0) = (d, d, 0) \text{ and } \gamma'(0) = \frac{1}{\sqrt{2}}(-1, 1, 0).$$

We have used the following code in Mathematica to compute solutions of this initial value problem:

```
H = 1

d = 0.88256

s = NDSolve[{2*H == ((x[t]*y'[t] - x'[t]*y[t] +
(x[t]^2 - y[t]^2)*(x'[t]*y[t] + x[t]*y'[t]))*(x'[t]^2
+ y'[t]^2) + 2*(x[t]*x'[t] + y[t]*y'[t])*(x[t]*y'[t]
+ x'[t]*y[t])*(y[t]*y'[t] - x[t]*x'[t])
+ (x[t]^2 + 1 + y[t]^2)*(x'[t]*y''[t] - x''[t]*y'[t]
-(y'[t]^2 - x'[t]^2)*(x'[t]*y[t] + x[t]*y'[t])))/(x'[t]^2
+ y'[t]^2 + (x'[t]*y[t] + x[t]*y'[t])^2)^(3/2),
x'[t]^2 + y'[t]^2 == 1, x[0] == d, y[0] == d,
x'[0] == -1/Sqrt[2], y'[0] == 1/Sqrt[2]}, {x, y},
{t, -3000, 3000}, AccuracyGoal -> Automatic,
MaxSteps -> 250000]

f = x /. First[First[s]]

g = y /. Last[Last[s]]

dom = First[Last[First[First[s]]]]

a = First[First[dom]]

b = Last[First[dom]]

ParametricPlot[{{f[t], g[t]}}, {t, 0, 8*4.4677}]
```

APPENDIX B

Minimax principle for minimal annuli

This appendix is the result of notes taken when checking the details of the minimax principle for minimal annuli by Ji Min [**Min93a**]. The results are from [**Min93a, Min89, MW93b**] and we state explicitly where they are from. We omit some proofs, give more detailed versions of other proofs and sometimes have to come up with a proof on our own.

In [**Min93a**] Ji Min obtained a multiple solution theorem for the Douglas problem of finding a minimal annulus bounded by two curves in a simply connected compact Riemannian 3-manifold. A perturbation method as in [**Uhl81**] is used and applied to an index generalising the Lusternik-Schnirelman category. This index is then a lower bound for the number of minimal annuli spanned by these two curves. In Chapter 5 we consider the case of two horizontal geodesics in $\mathbb{S}^3(\kappa,\tau)$ and show that the index is at least 2.

Let M denote a simply connected compact 3-manifold and let Γ_1 and Γ_2 be closed, rectifiable Jordan curves in M. We endow M with a Riemannian metric g such that (M,g) is isometrically embedded in $\mathbb{R}^k$ for some $k \in \mathbb{N}$, the curves Γ_1 and Γ_2 are geodesics in (M,g), which is possible since M is compact, and they are a positive distance apart. In this setting, $\exp_p\colon T_pM \to M$ maps $T_p\Gamma_j$ to Γ_j when $p \in \Gamma_j$ for $j \in \{1,2\}$.

Furthermore we will use the following notations:

$$\begin{aligned}
D &:= \{(x,y) \in \mathbb{R}^2 \colon x^2+y^2<1\},\\
A_\rho &:= A_{\rho,1} := \{(x,y) \in \mathbb{R}^2 \colon \rho < x^2+y^2<1\},\\
C_\rho &:= \{(x,y) \in \mathbb{R}^2 \colon x^2+y^2=\rho\}.
\end{aligned}$$

A minimal surface of annulus type bounded by Γ_1 and Γ_2 is characterised by a map $u \in C^2(A_\rho,M) \cap C^0(\overline{A_\rho},M)$ for some $\rho \in (0,1)$ such that the following is satisfied:

(MA1) $\Delta u = A(u)(du,du)$ in A_ρ,

(MA2) $|u_x|^2-|u_y|^2 = 0 = u_x \cdot u_y$ in A_ρ,

(MA3) $u|_{C_\rho}$ and $u|_{\partial D}$ are weakly monotone parametrisations of Γ_1 and Γ_2, respectively.

Here $A(u)$ denotes the second fundamental form of M in $\mathbb{R}^k$. The first property describes the harmonicity of u, the second one the conformity and the third one is a boundary condition. We note that u may have branch points.

Remark B.1. We note that a weakly monotone parametrisation assumes an orientation of Γ_1 and Γ_2. So, there are two choices to be made for Γ_2 once we have fixed an orientation of Γ_1. This is important for the geometry of the solution when we specialise to $\mathbb{S}^3(\kappa, \tau)$. In general, not every choice of orientations of the boundary curves admits a solution, for example coaxial circles with opposite orientations in $\mathbb{R}^3$.

B.1. Variational approach

As in the classical solution of the Plateau problem, we capture (MA1) to (MA3) as critical points of a suitable functional. First we introduce function spaces and some other notions.

For $p > 2$ the Sobolev space $W^{1,p}(A_\rho, \mathbb{R}^k)$ is embedded in $C^0(A_\rho, \mathbb{R}^k)$, so that the following definition makes sense:

Definition. For $p > 2$ we consider the spaces

$$W^{1,p}_{\Gamma_1,\Gamma_2}(A_\rho, M) := \big\{u \in W^{1,p}(A_\rho, \mathbb{R}^k)\colon u(A_\rho) \subset M, u(C_\rho) \subset \Gamma_1, u(\partial D) \subset \Gamma_2 \text{ and } \deg(u|_{C_\rho}) = 1 = \deg(u|_{\partial D})\big\}\,, \tag{B.1}$$

$$X(\rho) := X_p(\rho) := \Big\{u \in W^{1,p}_{\Gamma_1,\Gamma_2}(A_\rho, M)\colon u \text{ satisfies (MA3)}\Big\}\,. \tag{B.2}$$

Remark B.2. The space $W^{1,p}_{\Gamma_1,\Gamma_2}(A_\rho, M)$ is a Banach manifold. At $u \in W^{1,p}_{\Gamma_1,\Gamma_2}(A_\rho, M)$ we have the tangent space

$$T_u W^{1,p}_{\Gamma_1,\Gamma_2}(A_\rho, M) = \big\{v \in W^{1,p}(A_\rho, \mathbb{R}^k)\colon v(r,\theta) \in T_{u(r,\theta)}M \text{ for all } r \in (\rho, 1), v(\rho,\theta) \in T_{u(\rho,\theta)}\Gamma_1 \text{ and } v(1,\theta) \in T_{u(1,\theta)}\Gamma_2\big\}\,.$$

For $u \in W^{1,p}_{\Gamma_1,\Gamma_2}(A_\rho, M)$ and $v \in T_u W^{1,p}_{\Gamma_1,\Gamma_2}(A_\rho, M)$ we define the exponential map $e\colon TW^{1,p}_{\Gamma_1,\Gamma_2}(A_\rho, M) \to W^{1,p}_{\Gamma_1,\Gamma_2}(A_\rho, M)$ by $e(u,v) := \exp_{u(x,y)} v(x,y)$. We notice that $\exp(T\Gamma_j) \subset \Gamma_j$ since Γ_j is a geodesic in (M, g) for $j \in \{1, 2\}$.

We also note that $e(u,\cdot)\colon T_u W^{1,p}_{\Gamma_1,\Gamma_2}(A_\rho, M) \to W^{1,p}_{\Gamma_1,\Gamma_2}(A_\rho, M)$ is a local diffeomorphism and we denote its inverse by $e^{-1}(u,\cdot)$.

From now on we will fix $\rho_0 \in (0,1)$ and denote $X(\rho_0)$ by X. The space X does not have a manifold structure due to the boundary condition. We restrict the tangent space to X as follows:

Definition. For $u \in X$ we set

$$\begin{aligned} S_u := S_u X &:= \{v \in T_u W^{1,p}_{\Gamma_1,\Gamma_2}(A_{\rho_0}, M)\colon \|v\|_{C^0} < r_0, e(u,v) \in X\} \\ &= \{e^{-1}(u,w)\colon w \in X \text{ and } \|d(u,w)\|_{C^0} \leq r_0\}. \end{aligned}$$

Here r_0 denotes the injectivity radius of (M,g).

It turns out that this set is convex in a Banach space:

Lemma B.3 ([**MW93b**, Lemma 0.2])**.** *For $u \in X$ the tangent set S_u is a convex subset in $T_u W^{1,p}_{\Gamma_1,\Gamma_2}$.*

Proof. Let $\lambda \in [0,1]$ and $v_1, v_2 \in S_u$. We need to show that $w_\lambda := e(u, \lambda v_1 + (1-\lambda)v_2)$ is in X, that is, w_λ is monotone on C_{ρ_0} and ∂D. It suffices to prove this for ∂D because the proof for C_{ρ_0} is the same. We will let $u(\theta)$ and $w_\lambda(\theta)$ denote $u(1,\theta)$ and $w_\lambda(1,\theta)$, respectively. For any $\theta_0 \in [0, 2\pi]$, let U_0 be a neighbourhood of $u_0 := u(\theta_0)$ in Γ_2, and let $\psi\colon U_0 \to \mathbb{R}$ be the orientation preserving parametric representation which satisfies $\psi(u_0) = 0$ and $|\psi(w)| = d(u_0, w)$ for all $w \in U_0$. Let $\vec{v}$ be the unit section of $T\Gamma_2$ with the same orientation. Then we have

$$\begin{aligned} \psi(w_\lambda(\theta_0)) &= \psi(u(\theta)) + [\lambda v_1(\theta) + (1-\lambda)v_2(\theta)] \cdot \vec{v}(u(\theta)) \\ &= \lambda\psi(\exp_{u(\theta)} v_1(\theta)) + (1-\lambda)\psi(\exp_{u(\theta)} v_2(\theta)) \end{aligned}$$

for all θ close to θ_0. Since v_1 and v_2 are in S_u we see that $\psi(w_\lambda(\theta))$ is monotonic with respect to θ. $\square$

Thus we can study slopes of functionals over S_u.

Definition. Let $f\colon W^{1,p}_{\Gamma_1,\Gamma_2}(A_{\rho_0}, M) \times (0,1) \to \mathbb{R}$ be a C^2-functional.

(a) For $(u,\rho) \in X \times (0,1)$ we define $\beta\colon X \times (0,1) \to [0,\infty)$ by

$$\begin{aligned} \beta_1(u,\rho) &:= \sup\left\{\max(0, -\langle df(u,\rho), v\rangle)\colon v \in S_u, \|v\|_{W^{1,p}} \leq \frac{r_0}{2}\right\}, \\ \beta_2(u,\rho) &:= \left|\frac{\partial f(u,\rho)}{\partial \rho}\right|, \\ \beta(u,\rho) &:= \beta_1(u,\rho) + \beta_2(u,\rho). \end{aligned}$$

(b) We say that $(u,\rho) \in X \times (0,1)$ is *critical* with respect to $X \times (0,1)$ if $\beta(u,\rho) = 0$.
(c) The functional f satisfies the *Palais-Smale condition* with respect to $X \times (0,1)$ if for every sequence $\big((u_n,\rho_n)\big)_{n\in\mathbb{N}}$ in $X \times (0,1)$ such that $(f(u_n,\rho_n))_{n\in\mathbb{N}}$ is uniformly bounded and $\lim_{n\to\infty} \beta(f,u_n,\rho_n) = 0$, there exists a subsequence that converges strongly in $X \times (0,1)$.

In this setting we also have the following classical result:

Lemma B.4 ([**MW93b**, Lemma 0.4 to Lemma 0.6]). *Let*

$$f \colon W^{1,p}_{\Gamma_1,\Gamma_2}(A_{\rho_0}, M) \times (0,1) \to \mathbb{R}$$

be a C^2-functional.

(i) Set $K(f) := \{(u,\rho) \colon \beta(f,u,\rho) = 0\}$. There exists a bounded locally Lipschitz vector field V on $X \times (0,1) \setminus K(f)$ such that $V(u,\rho) \in S_u \times (0,1)$ and

$$\langle V(u,\rho)|_X, df(u,\rho)\rangle \le -\frac{1}{2}\beta(f,u,\rho)$$

for all $(u,\rho) \in X \times (0,1) \setminus K(f)$.

(ii) If f satisfies the Palais-Smale condition and there are no critical values on $[a,b]$ for $a < b$ then f_a is a strong deformation retract of f_b.

The proof is as in [**Pal66a**, Theorem 5.10].

We consider $(u,\rho) \in W^{1,p}_{\Gamma_1,\Gamma_2}(A_{\rho_0}, M) \times (0,1)$ and set

$$u^\rho(r,\theta) := u\left(\frac{\rho_0 - \rho + (1-\rho_0)r}{1-\rho}, \theta\right). \tag{B.3}$$

Then we have $u^\rho \in W^{1,p}_{\Gamma_1,\Gamma_2}(A_\rho, M)$. Furthermore we define the functional

$$F(u,\rho) := E(u^\rho) = \frac{1}{2}\int_{A_\rho} |\nabla u^\rho|^2 \, dx\, dy.$$

The following theorem is the starting point of this variational approach since critical points of F are minimal annuli bounded by Γ_1 and Γ_2.

Theorem B.5 ([**Min93a**, Theorem 1.3]). *Let $(u,\rho) \in X \times (0,1)$ be a critical point of the functional F. Then u^ρ satisfies (MA1) and (MA2) in A_ρ, that is, u^ρ parametrises a minimal annulus bounded by Γ_1 and Γ_2.*

Proof. We have to show that a critical point $(u, \rho) \in X \times (0, 1)$ satisfies (MA1) and (MA2) in A_ρ. For the proof of (MA1) we note that for a critical point (u, ρ) of F the function $u^\rho \in X(\rho)$ is a critical point of E. It is well-known that critical points of the Dirichlet energy are harmonic. Hence u^ρ is harmonic on A_ρ.

It remains to prove the conformity condition (MA2). To do so we will consider two types of variations: Tangential variations on the boundary and scalings of the inner radius.

Let $f_t \colon A_\rho \to A_\rho$ be a family of diffeomorphisms such that $f_0 = \mathrm{id}$ and $f_t|_{\partial D}$ as well as $f_t|_{C_\rho}$ are monotone for $t \in [0, \delta_0]$. Then $w_t := u^\rho \circ f_t \in X(\rho)$ and, if we write

$$f_t(x, y) = (\xi(x, y, t), \eta(x, y, t)), \quad \lambda = \frac{\partial \xi}{\partial t}|_{t=0}, \quad \mu = \frac{\partial \eta}{\partial t}|_{t=0},$$

we obtain

$$0 = \frac{d}{dt} E(w_t)|_{t=0} = \frac{1}{2} \int_{A_\rho} (|u_x^\rho|^2 - |u_y^\rho|^2)(\lambda_x - \mu_y) + 2\langle u_x^\rho, u_y^\rho \rangle (\lambda_y + \mu_x) \, dx \, dy.$$

Let $\Phi := |u_x^\rho|^2 - |u_y^\rho|^2 - i2\langle u_x^\rho, u_y^\rho \rangle$. Then Φ is analytic on A_ρ since u^ρ is harmonic. An application of the divergence theorem yields

$$\lim_{\delta \to 0} \operatorname{Im} \int_{\partial A_{\rho+\delta, 1-\delta}} (\lambda + i\mu) \Phi(z) \, dz = 0.$$

We specialise this variation now. Let $\alpha \in C^1 \left(\overline{A_\rho}, \mathbb{R} \right)$ be any continuously differentiable real function on $\overline{A_\rho}$ and set $f_t(z) := z e^{it\alpha(z)}$. Then, for sufficiently small $\delta_0 > 0$, this family restricted to $t \in [0, \delta_0]$ is an inner variation with $\lambda + i\mu = izf(r, \theta)$ and $dz = izd\theta$. Inserting this into the equation above we get

$$\begin{aligned} 0 &= \lim_{\delta \to 0} \left(\operatorname{Im} \int_0^{2\pi} \alpha(1 - \delta, \theta) z^2 \Phi(z) \, d\theta - \operatorname{Im} \int_0^{2\pi} \alpha(\rho + \delta, \theta) z^2 \Phi(z) \, d\theta \right) \\ &= \lim_{\delta \to 0} \left(\int_0^{2\pi} \alpha(1 - \delta, \theta) \operatorname{Im}(z^2 \Phi(z)) \, d\theta - \int_0^{2\pi} \alpha(\rho + \delta, \theta) \operatorname{Im}(z^2 \Phi(z)) \, d\theta \right). \end{aligned}$$

We can use cut-off functions to separate these two integrals and get

$$\lim_{r \to 1} \int_0^{2\pi} \alpha(r, \theta) \operatorname{Im}(z^2 \Phi(z)) \, d\theta = 0 = \lim_{r \to \rho} \int_0^{2\pi} \alpha(r, \theta) \operatorname{Im}(z^2 \Phi(z)) \, d\theta.$$

We show that $H(r, \theta) := \operatorname{Im}(z^2 \Phi(z))$ vanishes on A_ρ. We have

$$\lim_{r \to 1} \int_0^{2\pi} e^{in\theta} H(r, \theta) \, d\theta = 0 = \lim_{r \to \rho} \int_0^{2\pi} e^{in\theta} H(r, \theta) \, d\theta, \qquad \text{for all } n \in \mathbb{Z},$$

and define

$$K_n\colon (\rho,1)\to\mathbb{C},\qquad K_n(r):=\int_0^{2\pi} H(r,\theta)e^{in\theta}\,d\theta,\qquad n\in\mathbb{Z}.$$

We know that H is harmonic since it is the imaginary part of a holomorphic function, that is, $\Delta H = 0$ on A_ρ. Thus K_n satisfies the differential equation

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{n^2}{r^2}\right)K_n(r)=0.$$

The solutions of this ODE are $K_0(r) = a_0 + b_0\log(r)$ and $K_n(r) = a_n r^n + b_n r^{-n}$ for $n\in\mathbb{Z}\setminus\{0\}$, where a_n and b_n are complex constants. From the boundary condition we conclude $a_n = 0 = b_n$ for all $n\in\mathbb{Z}$ and thus $K_n\equiv 0$ on $(\rho,1)$. Hence the Fourier coefficients of H vanish and thus $H\equiv 0$ on A_ρ. The Cauchy-Riemann equations imply that $z^2\Phi(z)$ is a real constant.

Up to this point the proof is as in the case of the disc-type problem, where it is sufficient to consider the first type of variations. To finally show that $\Phi\equiv 0$ on A_ρ we also have to consider variations of the inner radius. Therefore we define

$$g_\rho^\sigma\colon A_\rho\to A_\sigma,\qquad g_\rho^\sigma(r,\theta):=\left(\frac{\sigma-\rho+(1-\sigma)r}{1-\rho},\theta\right)$$

for $0<\rho,\sigma<1$. We will abuse notation and identify the first component with g_ρ^σ. The inverse map of g_ρ^σ is g_σ^ρ. We compute

$$\begin{aligned}
&\frac{d}{d\sigma}\bigg|_{\sigma=\rho} 2E(u\circ g_\sigma^\rho) = \frac{d}{d\sigma}\bigg|_{\sigma=\rho}\int_0^{2\pi}\int_\sigma^1 |\partial_r(u\circ g_\sigma^\rho)|^2+\frac{1}{r^2}|\partial_\theta(u\circ g_\sigma^\rho)|^2\,d\,dr\,d\theta\\
&=\frac{d}{d\sigma}\bigg|_{\sigma=\rho}\int_0^{2\pi}\int_\sigma^1 |\partial_r u\circ g_\sigma^\rho|^2\cdot\left(\frac{1-\rho}{1-\sigma}\right)^2+\frac{1}{r^2}|\partial_\theta u\circ g_\sigma^\rho|^2\,r\,dr\,d\theta\\
&=\frac{d}{d\sigma}\bigg|_{\sigma=\rho}\int_0^{2\pi}\int_\rho^1\left(|\partial_r u|^2\left(\frac{1-\rho}{1-\sigma}\right)^2+\frac{1}{(g_\rho^\sigma)^2}|\partial_\theta u|^2\right)g_\rho^\sigma\cdot\frac{1-\sigma}{1-\rho}\,ds\,d\theta\\
&=\frac{d}{d\sigma}\bigg|_{\sigma=\rho}\int_0^{2\pi}\int_\rho^1|\partial_r u|^2\frac{1-\rho}{1-\sigma}g_\rho^\sigma+\frac{1}{\frac{1-\rho}{1-\sigma}g_\rho^\sigma}|\partial_\theta u|^2\,ds\,d\theta\\
&=\frac{1}{1-\rho}\int_0^{2\pi}\int_\rho^1|\partial_r u|^2-\frac{1}{r^2}|\partial_\theta u|^2\,dr\,d\theta.
\end{aligned}$$

In polar coordinates we have $z^2\Phi(z) = r^2|u_r|^2 - |u_\theta|^2 - 2ir\langle u_r, u_\theta\rangle$. Since $z^2\Phi(z)$ is a real constant we get

$$r^2|u_r|^2 - |u_\theta|^2 = c \qquad \text{on } A_\rho.$$

For a critical point (u,ρ) the derivative above vanishes and we obtain

$$\begin{aligned}
0 = \frac{d}{d\sigma}\bigg|_{\sigma=\rho} E(u\circ g_\sigma^\rho) &= \frac{1}{2}\frac{1}{1-\rho}\int_0^{2\pi}\int_\rho^1 |\partial_r u|^2 - \frac{1}{r^2}|\partial_\theta u|^2\,dr\,d\theta \\
&= \frac{1}{2}\frac{1}{1-\rho}\int_0^{2\pi}\int_\rho^1 \frac{1}{r^2}\left(r^2|\partial_r u|^2 - |\partial_\theta u|^2\right)\,dr\,d\theta \\
&= \frac{c}{2}\frac{2\pi}{1-\rho}\int_\rho^1 \frac{1}{r^2}\,dr = \frac{c\pi}{1-\rho}\left(-1+\frac{1}{\rho}\right) = \frac{c\pi}{\rho}.
\end{aligned}$$

This can only be true for $c = 0$. □

B.2. Perturbed functionals

The functional F does not seem to satisfy the Palais-Smale condition. We will perturb F to overcome this difficulty. For $\varepsilon \geq 0$ we define

$$F^\varepsilon := F + \varepsilon G,$$

where

$$G(u,\rho) := \frac{1}{p}\int_{A_\rho} |\nabla u^\rho|^p\,dx\,dy, \qquad (u,\rho) \in W^{1,p}_{\Gamma_1,\Gamma_2}(A_{\rho_0}, M)\times(0,1).$$

This approach has been introduced first by Uhlenbeck in [**Uhl81**]. The goal is to show that F and G satisfy the following properties, where the constants d^*, m^* and s_0 in (A4) are to be defined later:

- (A1) F is bounded from below, $G \geq 0$ and $\|dG\|$ is bounded on any set where G is bounded.
- (A2) $F + \varepsilon G$ satisfies the Palais-Smale condition for any $\varepsilon > 0$.
- (A3) If (u_n,ρ_n) is a sequence in $X\times(0,1)$ such that $F(u_n,\rho_n) \to c$, $\beta(F,u_n,\rho_n) \to 0$ and $G(u_n,\rho_n)$ is bounded, then c is a critical value of F.
- (A4) There exists $\varepsilon_0 > 0$ with the property that for any $b < \min(d^*, m^*+s_0)$ there exists $\alpha = \alpha(b)$ such that if (u,ρ) is a critical point of F^ε in $X\times(0,1)$ with $\varepsilon \in [0,\varepsilon_0]$ and $F^\varepsilon(u,\rho) \leq b$ then $G(u,\rho) \leq \alpha$.

These properties are sufficient to imply a minimax principle. One more property of the yet to be defined index is needed to count the number of critical points. We note that (A1) is easy to check.

We skip the proof of the following regularity theorem:

Theorem B.6 ([**Min89**, Theorem 2.2 and Theorem 2.3])**.** *Let (u, ρ) be a critical point of F^ε with respect to $X \times (0, 1)$.*

(i) We have $u^\rho \in W^{2,2}(A_\rho)$ and $\|u^\rho\|_{W^{2,2}(A_\rho)} \leq C$, where the constant C depends only on ρ, $1-\rho$, $p-2$ and the values $G(u, \rho)$ and $F^\varepsilon(u, \rho)$.
(ii) We have $u^\rho \in C^2(A_\rho)$.

For $\varepsilon = 0$ this is a well-known statement about regularity of minimal surfaces. For small $\varepsilon > 0$ the proof presented in [**Uhl81**] can be applied for inner regularity. The general case is stated in [**Min89**], but the proof is essentially the one presented in [**MW93b**]. We skip this here, because in our application the boundary curves admit rotations of angle π, extending a boundary point into an inner point of another minimal surface. With the result of Uhlenbeck we then have regularity up to the boundary.

We are now preparing the proof of (A2) and (A3). The following lemma is a compactness result for the conformal parameter $\rho \in (0, 1)$.

Lemma B.7 ([**Min93a**, Lemma 2.1])**.** *Let S be a subset of $X \times (0, 1)$ on which G is bounded. Then there is $\delta > 0$ such that $S \subset X \times [\delta, 1-\delta]$.*

Proof. We show first that $S \subset X \times (0, 1-\delta]$ for some $\delta > 0$. Let $u \in X \cap C^1(A_{\rho_0})$ and $\rho(0, 1)$. Then we have

$$G(u, \rho) = \frac{1}{p} \int_0^{2\pi} \int_\rho^1 |\nabla u^\rho|^p \, r \, dr \, d\theta$$

and since $\theta \mapsto \int_\rho^1 |\nabla u^\rho(r, \theta)|^p \, r \, dr$ is continuous there is $\theta_0 \in [0, 2\pi)$ such that

$$\int_\rho^1 |\nabla u^\rho(r, \theta_0)|^p \, r \, dr \leq \frac{p}{2\pi} G(u, \rho).$$

We use the Hölder inequality and the fact $\frac{r}{\rho} \geq 1$ for $r \in [\rho, 1]$ to show the estimate

$$0 < \operatorname{dist}(\Gamma_1, \Gamma_2) \leq |u^\rho(1, \theta_0) - u^\rho(\rho, \theta_0)| \leq \int_\rho^1 |\nabla u^\rho(r, \theta_0)|\, dr$$
$$\leq \left(\frac{(1-\rho)^{p-1}}{\rho} \int_\rho^1 |\nabla u^\rho(r, \theta_0)|^p\, r\, dr \right)^{\frac{1}{p}} \leq \left(\frac{p(1-\rho)^{p-1}}{2\pi\rho} G(u, \rho) \right)^{\frac{1}{p}}.$$

Therefore we find $C > 0$ such that

$$(1-\rho)^{p-1} \geq \frac{C}{G(u, \rho)} \qquad \text{for all } (u, \rho) \in X \times (0, 1).$$

Thus $S \subset X \times (0, 1-\delta]$ for some $\delta > 0$.

It remains to prove $S \subset X \times [\delta, 1)$ for some other $\delta > 0$. If that were false there would exist a sequence $\big((u_n, \rho_n)\big)_{n \in \mathbb{N}}$ in S such that $G(u_n, \rho_n) \leq B$ and $\lim_{n \to \infty} \rho_n = 0$. We will show that this leads to a contradiction. We have $G(u_n, \rho_n) \leq B$ and $p > 2$, so for $\alpha := 1 - \frac{2}{p} \in (0, 1)$ the Sobolev inequality implies that the C^α-norms of $u_n^{\rho_n}$ are uniformly bounded. Then $\rho_n \to 0$ implies that the perimeter of C_{ρ_n} tends to 0 and in combination with the uniform boundedness we have that the oscillations of $u_n^{\rho_n}$ on C_{ρ_n} tend to 0, too. However, $u_n^{\rho_n} \colon C_{\rho_n} \to \Gamma_1$ is a weakly monotone parametrisation of a Jordan curve, so that the oscillations of $u_n^{\rho_n}$ are a non-zero constant independent of n, a contradiction. □

In order to prove (A2) the following inequalities are useful:

Lemma B.8 ([**MW93b**, Lemma 1.1])**.** *There is a constant μ such that for all $x, y \in \mathbb{R}^k$ we have*

$$\langle |x|^{p-2}x - |y|^{p-2}y, x - y \rangle \geq \mu |x-y|^p \tag{B.4}$$

and

$$\langle a|x|^{p-2}x - b|y|^{p-2}y, x - y \rangle$$
$$\geq \mu(a+b)(|x|^{p-2} + |y|^{p-2})|x-y|^2 - |a-b| \cdot |x-y|(|x|^{p-1} + |y|^{p-1})$$

for all $a, b \geq 0$.

Proof. We only prove the first inequality since there was no proof for it in [**MW93b**, Lemma 1.1]. We use the monotonicity of $[0, \infty) \to \mathbb{R}$, $t \mapsto t^\alpha$ for any $\alpha > 0$ and for

$\mu = \frac{1}{2^{p-1}}$ we have

$$\begin{aligned}
&\langle |x|^{p-2}x - |y|^{p-2}y, x-y\rangle \\
&= \frac{1}{2}\langle (|x|^{p-2} + |y|^{p-2})(x-y) + |x|^{p-2} - |y|^{p-2})(x+y), x-y\rangle \\
&= \frac{1}{2}(|x|^{p-2} + |y|^{p-2})|x-y|^2 + \frac{1}{2}(|x|^{p-2} - |y|^{p-2})(|x|^2 - |y|^2) \\
&\geq \frac{1}{2}(|x|^{p-2} + |y|^{p-2})|x-y|^2 \\
&\geq \frac{1}{2}\frac{1}{2^{p-2}}|x-y|^{p-2}|x-y|^2 = \mu|x-y|^p.
\end{aligned}$$

The second inequality can be proved similarly. □

We need to ensure that a critical sequence has a convergent subsequence:

Lemma B.9 ([**Min93a**, Lemma 2.2]). *Let $\big((u_n, \rho_n)\big)_{n\in\mathbb{N}}$ be a sequence in $X \times (0,1)$ satisfying $\beta(F^\varepsilon, u_n, \rho_n) \to 0$ as $n \to \infty$ and $G(u_n, \rho_n) \leq C$ for some $C > 0$. Then there exists a subsequence, denoted still by $\big((u_n, \rho_n)\big)$ such that $\rho_n \to \overline{\rho} \in (0,1)$ and*

$$\int_{A_{\rho_0}} \left(|\nabla u_n - \nabla u_m|^2 + \varepsilon|\nabla u_n - \nabla u_m|^p\right)\, dx\, dy \to 0 \quad \text{as } n, m \to \infty.$$

Sketch of proof. In view of the previous lemma we may assume $\overline{\rho} = \lim_{n\to\infty}\rho_n \in (0,1)$. We prove that there is a constant $C = C(\rho_0, p) > 0$ such that

$$G(u, \rho_n) \geq C\frac{\rho_n}{(1-\rho_n)^{p-1}}\int_{A_{\rho_0}} |\nabla u|^p\, dx\, dy. \tag{B.5}$$

For a moment we will denote by ρ any element of the sequence $(\rho_n)_{n\in\mathbb{N}}$. Recall the map

$$g_{\rho_0}^{\rho}\colon A_{\rho_0} \to A_\rho, \qquad (r,\theta) \mapsto \left(\frac{\rho - \rho_0 + (1-\rho)r}{1-\rho_0}, \theta\right).$$

For $r \in [\rho_0, 1]$ we then have $\rho r \leq \frac{\rho r}{r} \leq \frac{\rho - \rho_0 + (1-\rho)r}{1-\rho_0} \leq \frac{r}{\rho_0}$ and thus, identifying $g_{\rho_0}^{\rho}$ with its first component, we get

$$\begin{aligned}
G(u,\rho) &= \int_{A_\rho} \frac{1}{p}|\nabla u^\rho|^p\, dx\, dy = \int_{g_{\rho_0}^{\rho}(A_{\rho_0})} \frac{1}{p}|\nabla u^\rho|^p\, dx\, dy \\
&= \int_0^{2\pi}\int_{\rho_0}^1 \frac{1}{p}\left(\frac{(1-\rho_0)^2}{(1-\rho)^2}|\partial_r u|^2 + \frac{1}{g_{\rho_0}^{\rho}(r)}|\partial_\theta u|^2\right)^{\frac{p}{2}} g_{\rho_0}^{\rho}(r)\frac{1-\rho}{1-\rho_0}\, dr\, d\theta \\
&\geq C\frac{\rho}{(1-\rho)^{p-1}}\int_{A_{\rho_0}} |\nabla u|^p\, dx\, dy.
\end{aligned}$$

The inequality (B.5), $G(u_n, \rho_n) \leq C$ and $\overline{\rho} \in (0,1)$ imply $\|u_n\|_{1,p} \leq D$. Thus for a subsequence we have $\|u_n - u_m\|_{C^0(A_{\rho_0})} \to 0$ for $n, m \to \infty$. For n, m large enough we then have $e^{-1}(u_n, u_m) \in S_{u_n}$, $e^{-1}(u_m, u_n) \in S_{u_m}$ and their $W^{1,p}$-norms are uniformly bounded.

Now let e_j^{-1} stand for the differential of $e^{-1}(\cdot,\cdot)$ with respect to the j-th argument, where $j \in \{1, 2\}$. Because of $e_2^{-1}(u,v)|_{v=u} = \mathrm{id}$ and $e_1^{-1}(u,v)|_{u=v} = -\,\mathrm{id}$ we see that

$$\nabla e^{-1}(u,v) = (\nabla v - \nabla u) + (e_2^{-1}(u,v) - e_2^{-1}(v,v))\nabla v + (e_1^{-1}(u,v) - e_1^{-1}(u,u))\nabla u.$$

For the inner variation we have

$$\langle dF^{\varepsilon}(u,\rho), e^{-1}(u,v)\rangle = \int_{A_\rho} (1 + \varepsilon|\nabla u|^{p-2})\nabla u \cdot \nabla e^{-1}(u,v)\, dx\, dy.$$

With Lemma B.8 we get

$$\begin{aligned}
&\langle dF^{\varepsilon}(u_n,\rho_m), e^{-1}(u_n,u_m)\rangle + \langle dF^{\varepsilon}(u_m,\rho_m), e^{-1}(u_m,u_n)\rangle \\
&\to \int_{A_{\rho_m}} \left[(1+\varepsilon|\nabla u_n^{\rho_m}|^{p-2})\nabla u_n^{\rho_m} - (1+\varepsilon|\nabla u_m^{\rho_m}|^{p-2})\nabla u_n^{\rho_n}\right](\nabla u_m^{\rho_m} - \nabla u_n^{\rho_n})\, dx\, dy \\
&\leq -\mu \int_{A_{\rho_m}} \frac{1}{2}|\nabla_n^{\rho_n} - \nabla_m^{\rho_m}|^2 + \frac{\varepsilon}{p}|\nabla_n^{\rho_n} - \nabla_m^{\rho_m}|^p\, dx\, dy
\end{aligned}$$

This estimate, $\beta(F^{\varepsilon}, u_n, \rho_n) \to 0$ and (B.5) prove the lemma. □

Finally we can prove (A2), that is, we verify the Palais-Smale condition for F^{ε} with respect to $X \times (0,1)$.

Theorem B.10 ([**Min93a**, Theorem 2.3])**.** *Let $\varepsilon > 0$. Then $F^{\varepsilon}\colon X \times (0,1) \to \mathbb{R}$ satisfies the Palais-Smale condition.*

Proof. Let $\big((u_n,\rho_n)\big)_{n\in\mathbb{N}}$ be a sequence in $X \times (0,1)$ such that $F^{\varepsilon}(u_n,\rho_n)$ is uniformly bounded in n and $\lim_{n\to\infty} \beta(F^{\varepsilon}, u_n, \rho_n) = 0$. By Lemma B.9 there is a subsequence, still denoted by n, such that ρ_n converges to some $\rho \in (0,1)$. The Poincaré inequality and Lemma B.9 also imply that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $W^{1,p}_{\Gamma_1,\Gamma_2}$, that is, it has a limit u in $W^{1,p}_{\Gamma_1,\Gamma_2}$. Since $W^{1,p}_{\Gamma_1,\Gamma_2}$ is embedded into $C^0_{\Gamma_1,\Gamma_2}$ for $p > 2$ we see that the boundary values converge in C^0, hence $u \in X$. □

The proof of Lemma B.9 implies also (A3):

Theorem B.11 ([**Min93a**, Theorem 2.4])**.** *Let $(u_n, \rho_n) \in X \times (0,1)$ be a sequence such that $F(u_n, \rho_n) \to c$, $\beta(F, u_n, \rho_n) \to 0$ and $G(u_n, \rho_n)$ is bounded. Then c is a critical value of F.*

Proof. From the proof of the previous lemma we see that a subsequence u_n converges to u in $H^1 \cap C^0$ and $\rho_n \to \overline{\rho}$ in $(0,1)$. Thus $\lim_{n\to\infty} F(u_n, \rho_n) = F(u, \overline{\rho}) = c$ and $\beta(F, u_n, \rho_n)$ converges to $\beta(F, u, \overline{\rho})$ for $n \to \infty$. Hence $(u, \overline{\rho})$ is a critical point of F with value c. □

B.3. Deformation Lemmas

Ultimately we are interested in the critical points of F. In critical point theories the domain is always supposed to be complete, because otherwise the pseudo-gradient flow goes over the boundary. In our situation the domain $X \times (0,1)$ is not complete. We show that under the Douglas condition we are actually in $X \times [\delta, 1-\delta]$ for some $\delta > 0$. Furthermore we define the spaces $X = X_p$ also for $p = 2$. Since we do not have an embedding into a space of continuous functions for $p = 2$ we must modify the definition as follows compared to the case $p > 2$:

Definition. Let

$$\begin{aligned} Y_j &:= \{\varphi \in C^0(\mathbb{S}^1, \Gamma_j) \colon \varphi \text{ is weakly monotone}\}, \qquad j \in \{1,2\}, \\ W^{1,2}_{\varphi} &:= \{u \in W^{1,2}(D, \mathbb{R}^k) \colon u(D) \subset M \text{ and } u|_{\partial D} = \varphi \text{ almost everywhere}\}, \\ W^{1,2}_{\varphi,\psi} &:= \{u \in W^{1,2}(A_{\rho_0}, \mathbb{R}^k) \colon u(A_{\rho_0}) \subset M, u(\rho_0, \cdot) = \varphi, u(1, \cdot) = \psi \text{ a.e.}\}. \end{aligned}$$

Define

$$\begin{aligned} X_2(\Gamma_1, \Gamma_2) &:= \bigcup_{\varphi \in Y_1} \bigcup_{\psi \in Y_2} W^{1,2}_{\varphi,\psi}, \\ X_2(\Gamma_j) &:= \bigcup_{\varphi \in Y_j} W^{1,2}_{\varphi}, \qquad j \in \{1,2\}. \end{aligned}$$

We also set

$$\begin{aligned} m_j &= m_j(\Gamma_j) = \inf\{E(u) \colon u \in X_2(\Gamma_j)\}, \qquad j \in \{1,2\}, \\ m^* &= m^*(\Gamma_1, \Gamma_2) = \inf\{F(u,\rho) \colon (u,\rho) \in X_2(\Gamma_1, \Gamma_2) \times (0,1)\}, \\ d^* &= d^*(\Gamma_1, \Gamma_2) = m_1 + m_2. \end{aligned}$$

The values m_1 and m_2 are the areas of disc-type surfaces bounded by Γ_1 and Γ_2, respectively, minimising the Dirichlet energy, whereas m^* is the area of annulus-type surfaces bounded by Γ_1 and Γ_2 minimising the Dirichlet energy. In $\mathbb{R}^3$ the Douglas condition is satisfied if there is an annulus with area less than d^*. The Douglas condition guarantees that a minimising sequence of annuli for the Dirichlet energy does not split up into two minimal discs.

In other ambient manifolds a further condition is needed to prevent a minimising sequence from splitting up into a minimal annulus and a minimal sphere:

$$s_0 = s_0(M) = \begin{cases} \inf\{E(u) |\ u\colon \mathbb{S}^2 \to M \text{ is harmonic and non-constant}\}, \\ \infty \text{ if } M \text{ admits no minimal sphere.} \end{cases}$$

With these notations we state two theorems that we need in order to establish the deformation lemmas. The first one is about the compactness of ρ under the Douglas condition:

Theorem B.12 ([**Min89**, Theorem 3.2])**.** *Let $b < d^*$. Then there exists $\delta > 0$ such that*

$$F_b := \{(u, \rho) \in X \times (0, 1)\colon F(u, \rho) < b\} \subset X \times [\delta, 1 - \delta].$$

If $((u_n, \rho_n))_{n\in\mathbb{N}}$ is a sequence in F_b then the sequences of boundary values $(u_n|_{C_{\rho_0}})_{n\in\mathbb{N}}$ and $(u_n|_{\partial D})$ are compact in C^0.

Proof. First we will show that ρ is bounded away from 1. Let $d := \operatorname{dist}(\Gamma_1, \Gamma_2)$ and recall $d > 0$ by assumption. For $\theta_0 \in (0, 2\pi]$ we compute

$$\begin{aligned} d &\leq |u(1, \theta_0) - u(\rho, \theta_0)| = \left| \int_\rho^1 u_r(r, \theta_0)\, dr \right| \\ &\leq \int_\rho^1 |u_r(r, \theta_0)|\, dr = \int_\rho^1 \frac{1}{\sqrt{r}} \sqrt{r}\, |u_r(r, \theta_0)|\, dr \\ &\leq \left(\int_\rho^1 \frac{1}{r}\, dr \right)^{\frac{1}{2}} \left(\int_\rho^1 |u_r(r, \theta_0)|^2\, r\, dr \right)^{\frac{1}{2}} \\ &= (-\ln(\rho))^{\frac{1}{2}} \left(\int_\rho^1 |u_r(r, \theta_0)|^2\, r\, dr \right)^{\frac{1}{2}}. \end{aligned}$$

Using $-\ln(\rho) = \ln(1/\rho) \leq \frac{1}{\rho} - 1 = \frac{1-\rho}{\rho}$ we get

$$d^2 \frac{\rho}{1-\rho} \leq \int_\rho^1 |u_r(r,\theta_0)|^2 \, r \, dr \qquad \text{for all } \theta_0 \in (0, 2\pi].$$

Moreover we have

$$\begin{aligned} F(u,\rho) &= \frac{1}{2} \int_0^{2\pi} \int_\rho^1 |u_r|^2 + \frac{1}{r^2} |u_\theta|^2 \, r \, dr \, d\theta \\ &\geq \frac{1}{2} \int_0^{2\pi} \left(\int_\rho^1 |u_r|^2 \, r \, dr \right) d\theta \\ &\geq \frac{1}{2} \int_0^{2\pi} \left(d^2 \frac{\rho}{1-\rho} \right) d\theta = \pi d^2 \frac{\rho}{1-\rho}. \end{aligned}$$

Thus ρ is bounded away from 1.

To prove that ρ is bounded away from 0 we will use the fact that A_ρ is conformal to the cylinder $C_{-\ln(\rho)} := \{(s,\theta) \colon s \in (0, -\ln(\rho)) \text{ and } \theta \in \mathbb{S}^1\}$. This conformal transformation yields a function $\tilde{u}$ on $C_{-\ln(\rho)}$. Since the Dirichlet energy is invariant under conformal transformations, and using the mean value theorem for integration, we get

$$\begin{aligned} E(\tilde{u}, C_{-\ln(\rho)}) &= \frac{1}{2} \int_0^{2\pi} \int_0^{-\ln(\rho)} \left(|\tilde{u}_s|^2 + |\tilde{u}_\theta|^2 \right) ds \, d\theta \\ &\geq \frac{1}{2} \int_0^{2\pi} \left(\int_0^{-\ln(\rho)} |\tilde{u}_\theta|^2 \, ds \right) d\theta = \frac{-\ln(\rho)}{2} \int_0^{2\pi} |\tilde{u}_\theta(h,\theta)|^2 \, d\theta \end{aligned}$$

for some $h \in (0, -\ln(\rho))$.

We will cut the cylinder $C_{-\ln(\rho)}$ along h and obtain two cylinders

$$\begin{aligned} C_1 &:= \{(s,\theta) \colon s \in (0, h] \text{ and } \theta \in (0, 2\pi]\}, \\ C_2 &:= \{(s,\theta) \colon s \in [h, -\ln(\rho)) \text{ and } \theta \in (0, 2\pi]\}. \end{aligned}$$

We insert a copy of D into the cut circle $\{s = h\}$ of each of the two cylinders. The domains $D_j := C_j \cup D$ for $j \in \{1,2\}$ are homeomorphic to a disc. Moreover, $\tilde{u}|_{\partial D_j}$ is a parametrisation of Γ_j for $j \in \{1,2\}$. The estimate above allows us to consider the extended maps

$$\tilde{u}_j \colon D_j \to M, \qquad \tilde{u}_j = \begin{cases} \tilde{u}_B & \text{on } B, \\ \tilde{u} & \text{on } C_j. \end{cases}$$

Since $\tilde{u}_j|_{\partial D_j}$ are monotonic parametrisations of Γ_j for $j \in \{1,2\}$ we have

$$\begin{aligned} E(\tilde{u}_j) = E(\tilde{u}|_{C_j}) + E(\tilde{u}_B) &\leq E(\tilde{u}|_{C_j}) + \frac{1}{2}\int_0^{2\pi} |\partial_\theta \tilde{u}_B(h,\theta)|^2\, d\theta \\ &= E(\tilde{u}|_{C_j}) + \frac{1}{2}\int_0^{2\pi} |\partial_\theta \tilde{u}(h,\theta)|^2\, d\theta \leq E(\tilde{u}|_{C_j}) + \frac{b}{-\ln(\rho)} \end{aligned}$$

and thus

$$d^* \leq E(\tilde{u}_1) + E(\tilde{u}_2) \leq E(\tilde{u}) + \frac{2b}{-\ln(\rho)} = E(u,\rho) + \frac{2b}{-\ln(\rho)}.$$

For $\rho \to 0$ this contradicts the assumption $E(u,\rho) \leq b < d^*$.

For the second statement of the theorem we refer to the proof of [**Jos91**, Theorem 4.7.1]. In fact, already the first part is included in that proof. □

Property (A4) is the following theorem:

Theorem B.13 ([**Min89**, Theorem 3.3])**.** *There exists $\varepsilon_0 > 0$ with the property that for any $b < \min(d^*, m^* + s_0)$ there exists $\alpha = \alpha(b)$ such that if (u,ρ) is a critical point of F^ε in $X \times (0,1)$ with $\varepsilon \in [0,\varepsilon_0]$ and $F^\varepsilon(u,\rho) \leq b$ then $G(u,\rho) \leq \alpha$.*

We skip the proof since it is technically involved and lengthy. It is based on ideas from [**BC85**] and [**Din85**]. With these two theorems we can prove the following deformation lemma:

Lemma B.14 ([**Min93a**, Lemma 3.4])**.** *If $b < \min(d^*, m^* + s_0)$ is not a critical value of F on $X \times (0,1)$, then there exists $\varepsilon_0 > 0$ such that F_b^ε is a strong deformation retract of F_b for $\varepsilon \leq \varepsilon_0$.*

Proof. First we claim that there exists $\varepsilon_0 > 0$ such that b is not a critical value of F^ε for all $\varepsilon \in [0,\varepsilon_0]$. If this were false we would find a sequence of critical points (u_n,ρ_n) of $F^{\frac{1}{n}}$ for the critical value b. For n sufficiently large we would have $G(u_n,\rho_n) \leq \alpha(b)$ by (A4). Furthermore $\beta(F^{\frac{1}{n}}, u_n, \rho_n) = 0$ implied $\beta(F, u_n, \rho_n) = -\frac{1}{n}\beta(G, u_n, \rho_n) \to 0$. We also had $F(u_n,\rho_n) \to b$ and so by (A3) b were a critical value of F, a contradiction.

Following [**Uhl81**] we consider

$$H\colon F_b \to (0,\infty), \qquad H(u,\rho) := \frac{G(u,\rho)}{b - F(u,\rho)}.$$

Then H is a C^1-functional on F_b.

We first show that H satisfies the Palais-Smale condition: Let (u_n, ρ_n) be a sequence such that $H(u_n, \rho_n)$ is bounded (say by $C > 0$) and $\beta(H, u_n, \rho_n) \to 0$.

If $\liminf_{n\to\infty} H(u_n, \rho_n) = 0$ then for a subsequence (still denoted by n) we also have $G(u_n, \rho_n) \to 0$. Using (B.5) we obtain

$$\begin{aligned}\int_{A_{\rho_0}} |\nabla u_n - \nabla u_m|^p \, dx \, dy &\le \int_{A_{\rho_0}} |\nabla u_n|^p \, dx \, dy + \int_{A_{\rho_0}} |\nabla u_m|^p \, dx \, dy \\ &\le C[G(u_n, \rho_n) + G(u_m, \rho_m)] \to 0.\end{aligned}$$

The Poincaré inequality implies that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X, that is, a subsequence of (u_n, ρ_n) converges in $X \times (0, 1)$, which finishes this case.

Otherwise we have $q := \liminf_{n\to\infty} H(u_n, \rho_n) > 0$. We see

$$H(u_n, \rho_n) \le C \iff G(u_n, \rho_n) \le C(b - F(u_n, \rho_n)) \iff F^{\frac{1}{C}}(u_n, \rho_n) \le b$$

and, using the estimate $G(u_n, \rho_n) \le C(b - F(u_n, \rho_n))$, we also have

$$\begin{aligned}\beta(H, u_n, \rho_n) \to 0 &\iff \frac{\beta(G, u_n, \rho_n)(b - F(u_n, \rho_n)) + \beta(F, u_n, \rho_n)G(u_n, \rho_n)}{(b - F(u_n, \rho_n))^2} \to 0 \\ &\iff \frac{\left[\beta(G, u_n, \rho_n)\frac{1}{C} + \beta(F, u_n, \rho_n)\right] G(u_n, \rho_n)}{(b - F(u_n, \rho_n))^2} \to 0 \\ &\iff \frac{\beta(F^{\frac{1}{C}}, u_n, \rho_n)H(u_n, \rho_n)}{b - F(u_n, \rho_n)} \to 0 \\ &\iff F^{\frac{1}{C}}(u_n, \rho_n) \to 0.\end{aligned}$$

By (A2) $F^{\frac{1}{C}}$ satisfies the Palais-Smale condition and so we have a convergent subsequence also in this case.

Looking at the equivalencies above we notice the following property for $\varepsilon > 0$: The functional H has critical value $\frac{1}{\varepsilon}$ if and only if F^ε has critical value b. Thus H has no critical values in $[1/\varepsilon_0, \infty)$. If we can prove that the pseudo-gradient flow stays away from the boundary of $X \times (0, 1)$ we can apply the deformation lemma from Palais to the functional H. We note $H_\infty = F_b$ and $H_{1/\varepsilon} = F_b^\varepsilon$ to complete the proof.

Assume that $(u(t), \rho(t))$ for $t \in [0, t_0)$ is the pseudo-gradient flow of H. We obviously have $(u(t), \rho(t)) \in H_\infty = F_b$ for $t \in [0, t_0)$. Since $b < d^*$ by assumption there is $\delta > 0$ such that $F_b \subset X \times [\delta, 1 - \delta]$, a contradiction to $\lim_{t\to t_0} \rho(t) \in \{0, 1\}$. $\square$

In the same vein we also have the following deformation lemma:

Lemma B.15 ([**Min93a**, Lemma 3.5])**.** *Suppose that F has no critical values in $[a, b]$ for $b < \min(m^* + s_0, d^*)$. Then for ε sufficiently small, F_a^ε is a strong deformation retract of F_b.*

Proof. Let $(b_n)_{n\in\mathbb{N}}$ be a monotonically increasing sequence with $b_n \in (a, b)$ and $b_n \to b$. Then, reasoning as in Lemma B.14, there exists $\varepsilon_1 > 0$ such that F^ε has no critical values in $[a, b_1]$ for all $\varepsilon \in [0, \varepsilon_1]$.

By Lemma B.4 (ii) and (A2), we see that $F_a^{\varepsilon_1}$ is a strong deformation retract of $F_{b_1}^{\varepsilon_1}$. Similarly, we find $\varepsilon_2 \in (0, \varepsilon_1)$ such that $F_{b_1}^{\varepsilon_2}$ is a strong deformation retract of $F_{b_2}^{\varepsilon_2}$. Copying the proof of Lemma B.14 we see that $F_{b_1}^{\varepsilon_1}$ is a strong deformation retract of $F_{b_1}^{\varepsilon_2}$.

We thus obtain a monotonically decreasing sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\varepsilon_n > 0$ and $\lim_{n\to\infty} \varepsilon_n = 0$ such that

$$F_{b_n}^{\varepsilon_n} \text{ is a strong deformation retract of } F_{b_{n+1}}^{\varepsilon_{n+1}}.$$

We conclude that $F_a^{\varepsilon_1}$ is a strong deformation retract of $F_b = \bigcup_{n\in\mathbb{N}} F_{b_n}^{\varepsilon_n}$. □

B.4. Minimax principle

We introduce a topological index and establish a minimax principle for the energy functional with respect to this index. Thus we obtain a multiple solution theorem for minimal annuli bounded by Γ_1 and Γ_2 in M.

Let $Z := W^{1,p}_{\Gamma_1,\Gamma_2}$ and for $u \in Z$ define

$$[u] := \{u \circ e^{i\theta} \colon \theta \in (0, 2\pi]\}.$$

We say that $B \subset Z \times (0, 1)$ is *deformable* to $[u] \times \{\rho\}$ in Z for some $(u, \rho) \in Z \times (0, 1)$ if there exists a continuous map $\mathcal{H} \colon [0, 1] \times B \to Z \times (0, 1)$ such that

$$\mathcal{H}(0, \cdot) = \mathrm{id} \quad \text{and} \quad \mathcal{H}(1, B) \subset [u] \times \{\rho\}.$$

Definition. For $A \subset Z \times (0, 1)$ we set

$$\begin{aligned} i(A) := \inf \{&k \geq 0 \colon \text{there exist } k \text{ closed subsets } B_1, \dots, B_k \subset Z \times (0, 1) \\ &\text{such that } A \subset B_1 \cup \cdots \cup B_k \text{ and } B_j \text{ is deformable to } [u_j] \times \{\rho_j\} \text{ in } Z \times (0, 1) \\ &\text{where } (u_j, \rho_j) \in Z \times (0, 1) \text{ for } j = 1, \dots, k\}\,. \end{aligned}$$

We call $i(A)$ *index* of A in $Z \times (0, 1)$. To emphasise this dependence we sometimes denote $i(A)$ by $i(A, Z \times (0, 1))$.

The following basic properties of this index are almost the same as those of the category in the sense of Lusternik-Schnirelman:

Proposition B.16 ([**Min93a**, Theorem 4.2])**.** *Let* $A, B \subset Z \times (0,1)$.

(i) $i(A) = 0$ *if and only if* $A = \emptyset$.
(ii) *If* $A \subset B$ *then* $i(A) \leq i(B)$.
(iii) $i(A \cup B) \leq i(A) + i(B)$.
(iv) $i([u] \times \{\rho\}) = 1$ *for all* $(u, \rho) \in Z \times (0,1)$.
(v) *Let* $B \subset Z$ *be a closed set and suppose a continuous map* $h\colon [0,1] \times B \to Z$ *satisfies* $h(0, \cdot) = \mathrm{id}$. *Then for* $A \subset B \times (0,1)$ *we have* $i(A) \leq i(h(1, A))$.

Proof. (i) to (iv) are easy to verify. For (v) let $B_1, \ldots, B_k \subset Z \times (0,1)$ be deformable and $h(1, A) \subset B_1 \cup \ldots \cup B_k$. Then $A_j := h^{-1}(1, B_j)$ cover A and each one of them is deformable, hence $i(A) \leq i(h(1, A))$. □

We want to prove a Lusternik-Schnirelman type of theorem with respect to i instead of the Lusternik-Schnirelman category. We need certain continuity properties of i to prove such a theorem.

Lemma B.17 ([**MW93b**, proof from Lemma 4.5])**.** *For* $(u, \rho) \in Z \times (0,1)$ *there exists a closed neighbourhood* V *of* $[u] \times \{\rho\}$ *in* $Z \in (0,1)$ *such that* V *is deformable to* $[u] \times \{\rho\}$.

Proof. Because of $\deg(u|_{\partial D}) = 1 = \deg(u|_{C_{\rho_0}})$ we see that $[u]$ is homeomorphic to $\mathbb{S}$. Thus $[u]$ is an absolute neighbourhood retract (in short ANR). Since $[u]$ is closed in Z, which is a metrisable Banach manifold, there exists a neighbourhood U of $[u]$ and a continuous map $f\colon U \to [u]$ satisfying $f(w) = w$ for all $w \in [u]$. Also since $[u]$ is compact there exists a neighbourhood W of $[u]$ such that $W \subset U$ and $d(f(w), w) < r_0$ for all $w \in W$.

Now it is easy to define a deformation map:

$$\mathcal{H}\colon [0,1] \times W \times (0,1) \to Z \times (0,1), \quad \mathcal{H}(t, w, \rho) := (e(f(w), (1-t)e^{-1}(f(w), w)), \rho).$$

Obviously, $V := W \times (0,1)$ is deformed to $[u] \times \{\rho\}$ by $\mathcal{H}$. □

This lemma gives rise to the continuity of the index i:

Theorem B.18 ([**Min93a**, Theorem 4.3])**.** *For* $A \subset Z \times (0,1)$ *there exists a neighbourhood* N_A *of* A *in* $Z \times (0,1)$ *such that* $i(A) = i(N_A)$.

Proof. For $i(A) = \infty$ we choose $N_A = Z \times (0,1)$. In the case $i(A) = k < \infty$ we have $A \subset \bigcup_{j=1}^k B_j$ for some closed subsets $B_1, B_2, \ldots, B_k \subset Z \times (0,1)$ which are deformable to $[u_j] \times \{\rho_j\}$ for $j \in \{1, \ldots, k\}$, respectively. Then we have continuous maps $f_j \colon [0,1] \times B_j \to Z \times (0,1)$ satisfying $f_j(0, \cdot) = \mathrm{id}$; without loss of generality we have either $f_j(1, B_j) = [u_j] \times \{\rho_j\}$ or $f_j(1, B_j) = \{(u_j, \rho_j)\}$ (since a proper subset of $[u_j] \times \{\rho_j\}$ is contractible to $\{(u_j, \rho_j)\}$) for $j \in \{1, \ldots, k\}$. Since $Z \times (0,1)$ is metrisable we can apply [**Pal66a**, Theorem 6] to find extensions

$$\tilde{f}_j \colon [0,1] \times Z \times (0,1) \to Z \times (0,1)$$

satisfying $\tilde{f}_j(t, u, \rho) = f_j(t, u, \rho)$ for all $(t, u, \rho) \in (\{0\} \times Z) \cup ([0,1] \times B_j)$ and all $j \in \{1, \ldots, k\}$. For each j, if $f_j(1, B_j) = (u_j, \rho_j)$, choose a contractible closed neighbourhood V_j of (u_j, ρ_j); if $f_j(1, B_j) = [u_j] \times \{\rho_j\}$, by Lemma B.17, we may choose a closed neighbourhood V_j of $[u_j] \times \{\rho_j\}$ such that V_j is deformable to $[u_j] \times \{\rho_j\}$. Thus $U_j := \tilde{f}_j^{-1}(1, V_j)$ is a closed neighbourhood of B_j and is deformable to $[u_j] \times \{\rho_j\}$. Obviously, we have $U := \bigcup_{j=1}^k U_j \supset A$ and $i(U) \leq k$, thus $N_A := \mathrm{int}(U)$ is as required. $\square$

In order to prove the Lusternik-Schnirelman type of theorem the following continuity of the index i is useful:

Theorem B.19 ([**Min93a**, Theorem 4.4])**.** *Let $K \subset Z \times (0,1)$ be compact. There exists a neighbourhood V of K in $C^0_{\Gamma_1,\Gamma_2}(A_{\rho_0}, M) \times (0,1)$ such that for $U \subset Z \times (0,1)$ satisfying $K \subset U \subset V$ we have $i(U) = i(V)$.*

We need the following technical result to prove this theorem:

Lemma B.20 ([**MW93b**, Lemma 4.8])**.** *The inclusion $j \colon Z \times (0,1) \to C^0_{\Gamma_1,\Gamma_2} \times (0,1)$ is a homotopy equivalence and for $A \subset Z \times (0,1)$ we have the following properties:*

(i) $i(A, Z) \leq i(A, C^0_{\Gamma_1,\Gamma_2})$,

(ii) if A is compact in $Z \times (0,1)$ then $i(A, Z) = i(A, C^0_{\Gamma_1,\Gamma_2})$.

Proof of Theorem B.19. For $i(\cdot, C^0_{\Gamma_1,\Gamma_2})$ it is evident that the statements from Proposition B.16 and Theorem B.18 remain true, with Z replaced by $C^0_{\Gamma_1,\Gamma_2}$. Then we may choose a neighbourhood V of K in $C^0_{\Gamma_1,\Gamma_2}$ such that $i(K, C^0_{\Gamma_1,\Gamma_2}) = i(V, C^0_{\Gamma_1,\Gamma_2})$. By Lemma B.20 we have $i(U, Z) \leq i(U, C^0_{\Gamma_1,\Gamma_2})$ and $i(K, Z) = i(K, C^0_{\Gamma_1,\Gamma_2})$. Since

$K \subset U \subset V$, we have

$$i(K, C^0_{\Gamma_1,\Gamma_2}) = i(K, Z) \leq i(U, Z) \leq i(U, C^0_{\Gamma_1,\Gamma_2}) \leq i(V, C^0_{\Gamma_1,\Gamma_2}) = i(K, C^0_{\Gamma_1,\Gamma_2}),$$

which proves the theorem. $\square$

We do not include the proof of Lemma B.20.

The subsequent theorem is the key property of the index i that we need.

Theorem B.21 ([**Min93a**, Theorem 4.5])**.** *For $b, \lambda, \alpha > 0$ put*

$$\begin{aligned}
K_{b,\alpha} &:= \{(u,\rho) \in X \times (0,1)\colon F(u,\rho) = b,\ \beta(u,\rho) = 0,\ G(u,\rho) \leq \alpha\}, \\
N_{b,\lambda,\alpha} &:= \left\{(u,\rho) \in X \times (0,1)\colon |F(u,\rho) - b| < \lambda,\ \beta(u,\rho) < \sqrt{\lambda},\ G(u,\rho) < \alpha + \lambda\right\}, \\
N^*_{b,\lambda,\alpha} &= \begin{cases} \left\{(u,\rho) \in X \times (0,1)\colon \operatorname{dist}((u,\rho), N_{b,\lambda,\alpha}) < 2\sqrt{\lambda}\right\} & \text{if } N_{b,\lambda,\alpha} \neq \emptyset, \\ \emptyset & \text{if } N_{b,\lambda,\alpha} = \emptyset. \end{cases}
\end{aligned}$$

There exists $\lambda > 0$ such that

$$i(N^*_{b,\lambda,\alpha}) = i(K_{b,\alpha}).$$

Proof. By Theorem B.6 and Lemma B.7, the set $K_{b,\alpha}$ is compact in $Z \times (0,1)$, so we know there exists a neighbourhood V of $K_{b,\alpha}$ in $C^0_{\Gamma_1,\Gamma_1} \times (0,1)$ such that $i(K_{b,\alpha}) = i(U)$ for all U satisfying $K_{b,\alpha} \subset U \subset V \cap Z \times (0,1)$. Now it suffices to prove $N^*_{b,\lambda,\alpha} \subset V$ for λ sufficiently small. Assume by contradiction that there are sequences $(\lambda_n)_{n\in\mathbb{N}}$ and (u_n, ρ_n) such that $\lambda_n \to 0$ and $(u_n,\rho_n) \in N^*_{b,\lambda_n,\alpha}$, but $(u_n,\rho_n) \notin V$. By definition there exists $(v_n, \tilde{\rho}_n)$ in $X \times (0,1)$ satisfying

$$\|u_n - v_n\|_{1,p} + |\rho_n - \tilde{\rho}_n| \to 0,\ F(v_n,\tilde{\rho}_n) \to b,\ \beta(v_n,\tilde{\rho}_n) \to 0 \text{ and } G(v_n,\tilde{\rho}_n) \leq \alpha + \lambda_n.$$

A subsequence then converges to some $(v_0, \tilde{\rho}) \in K_{b,\alpha}$, which implies $(u_n,\rho_n) \in V$ for large n, a contradiction. $\square$

Since $X \times (0,1) \subset Z \times (0,1)$ we set

$$c_\ell := \inf\left\{\sup_A F\colon A \subset X \times (0,1) \text{ and } i(A, Z \times (0,1)) \geq \ell\right\}$$

where $\ell \in [1, i(X \times (0,1), Z \times (0,1))]$.

We want to prove the following Lusternik-Schnirelman type of theorem. It tells us under which conditions c_ℓ is a critical value of F and how many critical points with this critical value exist.

Theorem B.22 ([**Min93a**, Theorem 4.6]). *The following statements are true:*

(i) If $c_\ell < \min(m^* + s_0, d^*)$ then c_ℓ is a critical value of F.

(ii) If $c_\ell = c_{\ell+1} = \cdots = c_{\ell+m} = c < \min(m^* + s_0, d^*)$ and c is not a limit of critical values of F then $i(K_c) \geq m + 1$ where K_c denotes the set of critical points of F with value c.

The first statement of the theorem could have been proved earlier already and requires only the deformation lemmas. Let us assume (i) is true. If, say, c_1 to c_ℓ are all distinct, then we have ℓ minimal annuli. If some of these values are equal and that value is a limit point of critical values then F has infinitely many critical points. Finally, the last case is covered by (ii) of the theorem. Therefore we get the following corollary:

Corollary B.23 ([**Min93a**, Theorem 4.7]). *If*

$$c_\ell < \min(m^* + s_0, d^*) \text{ for some } \ell \in [1, i(X \times (0,1))],$$

then there exist ℓ minimal annuli with areas $c_1, c_2, \ldots, c_\ell$ bounded by Γ_1 and Γ_2 in M.

We will apply this corollary to our problem in $\mathbb{S}^3(\kappa, \tau)$ when Γ_1 and Γ_2 are horizontal geodesics.

Proof of Theorem B.22. First, let us summarise some principals we have proved thus far:

(A1) F is bounded from below, $G \geq 0$ and $\|dG\|$ is bounded on any set where G is bounded.

(A2) $F + \varepsilon G$ satisfies the Palais-Smale condition for any $\varepsilon > 0$.

(A3) If (u_n, ρ_n) is a sequence in $X \times (0,1)$ such that $F(u_n, \rho_n) \to c$, $\beta(F, u_n, \rho_n) \to 0$ and $G(u_n, \rho_n)$ is bounded, then c is a critical value of F.

(A4) There exists $\varepsilon_0 > 0$ with the property that for any $b < \min(d^*, m^* + s_0)$ there exists $\alpha = \alpha(b)$ such that if (u, ρ) is a critical point of F^ε in $X \times (0,1)$ with $\varepsilon \in [0, \varepsilon_0]$ and $F^\varepsilon(u, \rho) \leq b$ then $G(u, \rho) \leq \alpha$.

(A5) There exists $\lambda > 0$ such that $i(N^*_{b,\lambda,\alpha}) = i(K_{b,\alpha})$.

Suppose (i) were false. If c_ℓ is not a critical value of F, then (A3) and (A4) imply that F has no critical values in $[c_\ell - \delta, c_\ell + \delta]$ for some $\delta > 0$. By Lemma B.15, $F^\varepsilon_{c_\ell-\delta} \subset F_{c_\ell-\delta}$ is a strong deformation retract of $F_{c_\ell+\delta}$. Thus, if $A \subset F_{c_\ell+\delta}$ is a closed set with $i(A) \geq \ell$, then A is deformable to a set $B \subset F_{c_\ell-\delta}$. This is a contradiction to the definition of c_ℓ.

Now we prove (ii). Let $c_\ell = \cdots = c_{\ell+m} = c < \min(m^* + s_0, d^*)$ and assume that c is not a limit of critical values of F. Then there is $\delta > 0$ such that F has no critical values in $(c, c+\delta]$ and $c + 2\delta < \min(m^* + s_0, d^*)$. We find $\varepsilon > 0$ such that $F^\varepsilon_{c+\delta}$ is a strong deformation retract of $F_{c+\delta}$. Put $c^\varepsilon_j = c_j(F^\varepsilon)$. Then we have

$$c^\varepsilon_\ell \leq \cdots \leq c^\varepsilon_{\ell+m} \leq c + \delta$$

since $i(F^\varepsilon_{c+\delta}) \geq i(F_{c+\delta}) \geq \ell + m$.

Now let $\eta \in C^2(\mathbb{R}, \mathbb{R})$ satisfy

(a) $\eta(s) = 0$ for all $s \leq \alpha$,
(b) $0 < \eta'(s) \leq \varepsilon$ for all $s > \alpha$,
(c) $s\eta'(s) \leq \delta + \eta(s)$,
(d) $\lim_{s\to\infty} \eta(s) = +\infty$,

where $\alpha = \alpha(c + \delta)$ is determined by (A4).

We define a functional $J \in C^2(Z \times (0,1), \mathbb{R})$ by

$$J(u,\rho) := F(u,\rho) + \eta(G(u,\rho)), \qquad (u,\rho) \in Z \times (0,1).$$

We claim that

$$K_b(J) = K_{b,\alpha} \quad \text{ for all } b \leq c + \delta.$$

The inclusion $K_{b,\alpha} \subset K_b(J)$ is clear. Consider $(u,\rho) \in K_b(J)$ with $b \leq c + \delta$ and let $\gamma = \eta'(G(u,\rho))$. By (c) we have

$$\begin{aligned} F^\gamma(u,\rho) &=\leq F(u,\rho) + \eta(G(u,\rho)) + \delta = J(u,\rho) + \delta \leq c + 2\delta, \\ dF^\gamma(u,\rho) &= dF(u,\rho) + \gamma dG(u,\rho) = dJ(u,\rho), \end{aligned}$$

that is, $(u,\rho) \in K_{c+2\delta}(F^\gamma)$. If $\gamma > 0$ we get $G(u,\rho) \leq \alpha$ by (A4). By (a) we then have $\gamma = 0$, so indeed $(u,\rho) \in K_{b,\alpha}$.

We see that $F \leq J \leq F^\varepsilon$. So we have

$$c = c_j \leq c_j(J) \leq c^\varepsilon_j \leq c + \delta, \qquad \ell \leq j \leq \ell + m.$$

To finish the proof of this theorem we need one more lemma:

Lemma B.24 ([**MW93b**, Theorem 4.4]). *For $b \in \mathbb{R}$ and $\lambda > 0$ let*

$$N_{b,\lambda}(J) := \Big\{(u,\rho) \in X \times (0,1) \colon |J(u,\rho) - b| < \lambda \text{ and } \beta(J,u,\rho) < \sqrt{\lambda}\Big\}$$

as well as

$$N^*_{b,\lambda} := \begin{cases} \Big\{(u,\rho) \in X \times (0,1) \colon \operatorname{dist}((u,\rho), N_{b,\lambda}) < 2\sqrt{\lambda}\Big\} & \text{if } N_{b,\lambda} \neq \emptyset, \\ \emptyset & \text{if } N_{b,\lambda} = \emptyset. \end{cases}$$

(i) For every $b \in (-\infty, c+\delta]$ there is $\lambda > 0$ such that $i(K_b(J)) = i(N^*_{b,\lambda}(J)))$.

(ii) Each $c_j(J)$ is a critical value of J.

(iii) We have $c_j(J) = c$ for all j and $i(K_c(J)) \geq m+1$

Proof. Let us prove (i). Assume first that $K_b(J) = \emptyset$. Then there is $\lambda > 0$ such that $N^*_{b,\lambda}(J) = \emptyset = N_{b,\lambda}(J)$. If this were not the case then $N_{b,\lambda}(J) \neq \emptyset$ and we find a sequence $(u_n, \rho_n) \in X \times (0,1)$ such that

$$\lim_{n\to\infty} J(u_n,\rho_n) = b \quad \text{and} \quad \lim_{n\to 0} \beta(J,u_n,\rho_n) = 0.$$

Let $\gamma_n := \eta'(G(u_n,\rho_n))$. Then, for a subsequence, $\gamma = \lim_{n\to\infty} \gamma_n \geq 0$. By (c) and (d) we get

$$\begin{aligned} F^{\gamma_n}(u_n,\rho_n) &= F(u_n,\rho_n) + \gamma_n G(u_n,\rho_n) \\ &\leq F(u_n,\rho_n) + \eta(G(u_n,\rho_n)) + \delta \\ &= J(u_n,\rho_n) + \delta \end{aligned}$$

and $G(u_n,\rho_n)$ is bounded (hence $\beta(G,u_n,\rho_n)$ is bounded). For a subsequence we thus have

$$F^{\gamma}(u_n,\rho_n) \to b' \leq b + \delta \leq c + 2\delta \quad \text{and} \quad \beta(F^{\gamma},u_n,\rho_n) \to 0.$$

If $\gamma > 0$, then by (A2) and (A4), the sequence (u_n,ρ_n) converges to a critical point (u,ρ) of F^{γ} with $G(u,\rho) \leq \alpha$. By definition of η we have $\gamma = 0$ and thus

$$\lim_{n\to\infty} G(u_n,\rho_n) \leq \alpha.$$

Then we have $F(u_n,\rho_n) \to b$, $\beta(F,u_n,\rho_n) \to 0$, and from (A3) we derive that b is a critical value of F, a contradiction.

We consider the case $K_b(J) \neq \emptyset$ now. It suffices to show that $N^*_{b,\lambda} \subset N^*_{b,\overline{\lambda},\alpha}$ for some $\lambda > 0$ where $\overline{\lambda}$ is as in (A5). Assume this is false for every $\lambda > 0$. Then there are sequences $(\lambda_n)_{n\in\mathbb{N}}$ and $((u_n,\rho_n))_{n\in\mathbb{N}}$ satisfying $\lim_{n\to\infty}\lambda_n = 0$, $(u_j,\rho_n)\in N^*_{b,\lambda_n}$ but $(u_n,\rho_n)\notin N^*_{b,\overline{\lambda},\alpha}$ for all $n\in\mathbb{N}$. By definition of $N^*_{b,\overline{\lambda},\alpha}$ there is $(v_n,\overline{\rho}_n)\in X\times(0,1)$ such that $\|u_n - v_n\|_{1,p} + |\rho_n - \overline{\rho}_n| \to 0$, $J(v_n,\overline{\rho}_n)\to b$ and $\beta(J,v_n,\overline{\rho}_n)\to 0$. Arguing as in the previous case, we see that for a subsequence $F(v_n,\overline{\rho}_n)\to b$, $\beta(F,v_n,\overline{\rho}_n)\to 0$ and $\lim_{n\to\infty} G(v_n,\overline{\rho}_n)\le\alpha$. This shows that $v_n\in N_{b,\overline{\lambda},\alpha}$ and thus $u_n\in N^*_{b,\overline{\lambda},\alpha}$ for large n, a contradiction.

To prove (ii) we will use (i). We choose $\lambda > 0$ as in (i). Inspecting the proof of [**Pal66b**, Theorem 5.11] we see that $J_{c_j(J)+\lambda}\setminus N^*_{c_j(J),\lambda}$ can be deformed to a subset of $J_{c_j(J)-\lambda}$, hence

$$i(J_{c_j(J)+\lambda}\setminus N^*_{c_j(J),\lambda}) \le i(J_{c_j(J)-\lambda}).$$

Since the index i is sub-additive and using (i) we also get

$$i(J_{c_j(J)+\lambda}) \le i(J_{c_j(J)+\lambda}\setminus N^*_{c_j(J),\lambda}) + i(N^*_{c_j(J),\lambda}) \le i(J_{c_j(J)-\lambda}) + i(K_{c_j(J)}(J)).$$

By definition of $c_j(J)$ we have $i(J_{c_j(J)+\lambda})\ge j$ and $i(J_{c_j(J)-\lambda})\le j-1$. Combining these with the inequalities above we see that $i(K_{c_j(J)}(J))\ge 1$.

Finally we prove (iii). We know that a critical value of J is also one of F, so $c_j < c+\delta$ and thus $c_j(J) = c$. Arguing as in the proof of (ii) we see that $i(J_{c+\lambda})\ge \ell+m$ and $i(J_{c-\lambda})\ge \ell-1$. This shows $i(K_c(J))\ge m+1$. $\square$

We showed that $K_{c,\alpha} = K_c(J)$, so that Lemma B.24 (iii) finishes the proof. $\square$

Bibliography

[AR05] Uwe Abresch and Harold Rosenberg, *Generalized Hopf Differentials*, Mat. Contemp. **28** (2005), 1–28.

[BC85] Vieri Benci and Jean-Michel Coron, *The Dirichlet problem for harmonic maps from the disk into the euclidean n-sphere*, Annales de l'institut Henri Poincaré (C) Analyse non linéaire **2** (1985), no. 2, 119–141.

[BdCH09] Allen Back, Manfredo P. do Carmo, and Wu-Yi Hsiang, *On the fundamental equations of equivariant geometry*, Tamkang Journal of Mathematics **40** (2009), no. 4, 343–376.

[BL01] Haim Brezis and Yanyan Li, *Topology and Sobolev Spaces*, Journal of Functional Analysis **183** (2001), 321–369.

[Dan07] Benoît Daniel, *Isometric immersions into 3-dimensional homogeneous manifolds*, Commentarii Mathematici Helvetici **82** (2007), no. 1, 87–131.

[Del41] Charles-Eugène Delaunay, *Sur la surface de révolution dont la courbure moyenne est constante*, Journal de Mathématiques Pures et Appliquées **1** (1841), no. 6, 309–314.

[Des15] Christophe Desmonts, *Constructions of Periodic Minimal Surfaces and Minimal Annuli in* Sol_3, Pacific Journal of Mathematics **276** (2015), no. 1, 143–166.

[DH09] Benoît Daniel and Laurent Hauswirth, *Half-space theorem, embedded minimal annuli and minimal graphs in the heisenberg group*, Proc. London Math. Soc. **98** (2009), no. 3, 445–470.

[Din85] Weiyue Ding, *Lusternik-Schnirelmann Ttheory for Harmonic Maps*, Acta Mathematica Sinica, English Series **2** (1985), no. 2, 105–122.

[dLH10] Jorge de Lira and Jorge Hinojosa, *The gauss map of minimal surfaces in berger spheres*, Annals of Global Analysis and Geometry **37** (2010), no. 2, 143–162.

[DM13] Benoît Daniel and Pablo Mira, *Existence and uniqueness of constant mean curvature spheres in* Sol_3, Journal für die reine und angewandte Mathematik **685** (2013), 1–32.

[Eng06] Sven Engel, *On the geometry and trigonometry of homegeneous 3-manifolds with 4-dimensional isometry group*, Mathematische Zeitschrift **254** (2006), no. 3, 439–459.

[Esc89] Jost-Hinrich Eschenburg, *Maximum principle for hypersurfaces*, Manuscripta mathematica **64** (1989), 55–75.

[FMP99] Chrstiam B. Figueroa, Francesco Mercuri, and Renato H. L. Pedrosa, *Invariant Surfaces of the Heisenberg Groups*, Annali di Matematica pura ed applicata **CLXXVII** (1999), no. IV, 173–194.

[GB93] Karsten Große-Brauckmann, *New surfaces of constant mean curvature*, Mathematische Zeitschrift **214** (1993), no. 1, 527–565.

[GB05] ______, *Cousins of Constant Mean Curvature Surfaces*, Global Theory of Minimal Surfaces, vol. 2, Clay Mathematisc Proceedings, 2005, pp. 747–767.

[GBK12] Karsten Große-Brauckmann and Rob B. Kusner, *Conjugate Plateau Constructions for Homogeneous* 3*-Manifolds*, Preprint, 2012.

[Ger31] Sophie Germain, *Mémoire sur la courbure des surfaces*, Journal für die reine und angewandte Mathematik **7** (1831), no. 1, 1–29.

[HdLR05] David Hoffman, Jorge S.H. de Lira, and Harold Rosenberg, *Constant Mean Curvature Surfaces in* $M^2 \times \mathbb{R}$, Transactions of the American Mathematical Society **358** (2005), no. 2, 491–507.

[HH89] Wu-Teh Hsiang and Wu-Yi Hsiang, *On the uniqueness of isoperimetric solutions and imbedded soap bubbles in non-compact symmetric spaces*, Invent. math. **98** (1989), 39–58.

[Jos91] Jürgen Jost, *Two-Dimensional Geometric Variational Problems*, Pure and Applied Mathematics, John Wiley & Sons, 1991.

[Kil12] Martin Kilian, *Bubbletons are not embedded*, Osaka Journal Mathematics **49** (2012), no. 3, 653–663.

[KKMS92] Nicholas J. Korevaar, Robert Kusner, William H. Meeks, and Bruce Solomon, *Constant mean curvature surfaces in hyperbolic space*, American Journal of Mathematics **114** (1992), no. 1, 1–43.

[KKS89] Nicholas J. Korevaar, Robert Kusner, and Bruce Solomon, *The structure of complete embedded surfaces with constant mean curvature*, J. Differential Geometry **30** (1989), 465–503.

[LM14] Rafael Lopez and Marian Ioan Munteanu, *Invariant surfaces in the homogeneous space* Sol_3 *with constant curvature*, Mathematische Nachrichten **287** (2014), no. 8–9, 1013–1024.

[Lop14] Rafael Lopez, *Invariant surfaces in* Sol_3 *with constant mean curvature and their computer graphics*, Advances in Geometry **14** (2014), no. 1.

[Man14] José M. Manzano, *On the Classification of Killing Submersions and their Isometries*, Pacific Journal of Mathematics **270** (2014), no. 2, 367–392.

[Maz13] Laurent Mazet, *A general halfspace theorem for constant mean curvature surfaces*, American Journal of Mathematics **135** (2013), no. 3, 801–834.

[Maz15] ______, *Cylindrically bounded constant mean curvature surfaces in* $\mathbb{H}^2 \times \mathbb{R}$, Transactions of the American Mathematical Society **367** (2015), no. 8, 5329–5354.

[Mee13] William Meeks, *Constant Mean Curvature Spheres in* Sol_3, American Journal of Mathematics (2013), no. 135.

[Men13] Ana Menezes, *Periodic Minimal Surfaces in Semidirect Products*, Journal of the Australian Mathematical Society **96** (2013), no. 1, 127–144.

[Min89] Ji Min, *An a Priori Estimate for Douglas Problem in Riemannian Manifolds*, Acta Mathematica Sinica, English Series **5** (1989), no. 3, 235–249.

[Min93a] ______, *Minimal Annuli in Riemannian Manifolds*, Acta Mathematica Sinica, English Series **9** (1993), no. 1, 74–89.

[Min93b] ______, *Multiple Solutions to the Douglas Problem in* $\mathbb{S}^n$, SCIENCE CHINA Mathematics **36** (1993), no. 10, 1162–1168.

[MMPR13] W. H. Meeks, III, P. Mira, J. Perez, and A. Ros, *Constant mean curvature spheres in homogeneous three-spheres*, ArXiv e-prints (2013).

[MP12] William H. Meeks and Joaquin Perez, *Constant Mean Curvature Surfaces in metric lie groups*, Contemporay Mathematics **570** (2012), 25–110.

[MT14] José M. Manzano and Francisco Torralbo, *New Examples of Constant Mean Curvature in* $\mathbb{S}^2 \times \mathbb{R}$ *and* $\mathbb{H}^2 \times \mathbb{R}$, Michigan Math J. **63** (2014), no. 4, 701–723.

[MW93a] William H. Meeks and Brian White, *The Space of Minimal Annuli Bounded by an Extremal Pair of Planar Curves*, Communications in Analysis and Geometry **1** (1993), no. 3–4, 415–437.

[MW93b] Ji Min and Guang Yin Wang, *Minimal surfaces in Riemannian manifolds*, Memoirs of the American Mathematical Society **104** (1993), no. 495.

[Onn08] Irene I. Onnis, *Invariant surfaces with constant mean curvature in* $\mathbb{H}^2 \times \mathbb{R}$, Annali di Matematica **187** (2008), 667–682.

[Pal66a] Richard Palais, *Homotopy Ttheory of Iinfinite Dimensional Manifolds*, Topology **5** (1966), 1–16.

[Pal66b] ______, *Lusternik-Schnirelman Ttheory on Banach Manifolds*, Topology **5** (1966), 115–132.

[Pen10] Carlos Penafiel, *Surfaces of constant mean curvature in homogeneous three manifolds with emphasis in* $\mathrm{PSL}_2(\mathbb{R}, \tau)$, Ph.D. thesis, PUC-RJ, Brazil, 2010.

[Sch83] Richard M. Schoen, *Uniqueness, Symmetry and Embeddedness of Minimal Surfaces*, Journal of Differential Geometry **18** (1983), 791–809.

[ST05] Ricardo Sa Earp and Eric Toubiana, *Screw motion surfaces in* $\mathbb{H}^2 \times \mathbb{R}$ *and* $\mathbb{S}^2 \times \mathbb{R}$, Illinois J. Math **49** (2005), 1323–1362.

[Stu41] Jacques Charles François Sturm, *Note à l'occasion de l'article précédent*, Journal de Mathématiques Pures et Appliquées **1** (1841), no. 6, 315–320.

[SW93] Ian Sterling and Henry C. Wente, *Existence and Classification of Constant Mean Curvature Multi-bubbletons of Finite and Iinfinite Type*, Indiana University Mathematics Journal **42** (1993), no. 4, 1239–1266.

[Tor10] Francisco Torralbo, *Rotationally invariant constant mean curvature surfaces in homogeneous* 3*-manifolds*, Differential Geometry and its Applications **28** (2010), no. 5, 593–607.

[Tor12] ______, *Compact minimal surfaces in the Berger spheres*, Annals of Global Analysis and Geometry **41** (2012), no. 4, 391–405.

[TY90] Friedrich Tomi and Rugang Ye, *The exterior plateau problem*, Mathematische Zeitschrift **205** (1990), no. 1, 233–245.

[Uhl81] Karen Uhlenbeck, *Morse Theory by Perturbation Methods with Applications to Harmonic Maps*, Transactions of the American Mathematical Society **267** (1981), no. 2, 569–583.

[Vrz14] Miroslav Vrzina, *Cylinders as invariant cmc surfaces in simply connected homogeneous* 3*-manifolds*, ArXiv e-prints (2014).

Index

Akademischer Lebenslauf

Miroslav Vržina

10. Juni 1986	geboren in Offenbach am Main, Deutschland
1993 – 2006	Schulbesuch
Juni 2006	Abitur an der Claus-von-Stauffenberg-Schule, Rodgau, Deutschland
10/2006 – 3/2011	Diplomstudium Mathematik mit Nebenfach Informatik an der Technischen Universität Darmstadt
März 2011	Diplom in Mathematik
	Diplomarbeit “Ends of constant mean curvature in $\mathbb{H}^2 \times \mathbb{R}$” betreut durch Karsten Große-Brauckmann
seit April 2011	Wissenschaftlicher Mitarbeiter am Fachbereich Mathematik (Arbeitsgruppe Geometrie und Approximation) der Technischen Universität Darmstadt
5. Februar 2016	Einreichung der Dissertation
22. April 2016	Tag der mündlichen Prüfung